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Supplement to the paper "The density of the sample correlations under elliptical symmetry with or without the truncated variance-ratio"

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This article supplements Ogasawara (2022) using the same notations and definitions.

S.1 The density of the sample correlation matrix without truncation under elliptical symmetry

Lemma S1. *Let* $\mathbf{u} = (v_{11}, \ldots, v_{pp}, \mathbf{r}^{\mathrm{T}})^{\mathrm{T}} \sim \mathrm{W}_p(\boldsymbol{\rho}, n)$ *with* $n > 1$ *and* $\mathbf{P} > 0$. *Then, an integral expression of the marginal joint pdf of* $\mathbf{r}$ *or* $\mathbf{R}$ *is*

$$f_p(\mathbf{r} \mid \boldsymbol{\rho}, n) = f_p(\mathbf{R} \mid \boldsymbol{\rho}, n)$$
$$= C_p \mid \mathbf{R} \mid^{(n-p-1)/2} \int_{\mathbf{0}}^{\infty} \prod^{i(1,\ldots,p)} \exp\left(-\frac{\rho^{ii} v_{ii}}{2} - g_i v_{ii}^{1/2} \right) v_{ii}^{(n-2)/2} \mathrm{d}v_{ii},$$

where $1 / C_p = 2^{np/2} \mid \mathbf{P} \mid^{n/2} \Gamma_p(n/2)$ *is the normalizing constant for the Wishart distribution shown in the main paper when* $\boldsymbol{\Sigma} = \mathbf{P}$, $\mathbf{P}^{-1} = \{\rho^{ij}\}$;

$$g_i = g_i(\rho_{i,i+1}, \ldots, \rho_{ip}, r_{i,i+1}, \ldots, r_{ip}, v_{i+1,i+1}, \ldots, v_{pp}) = \sum\nolimits_{j=i+1}^{p} \rho^{ij} r_{ij} v_{jj}^{1/2}$$

$(i = 1, \ldots, p)$, $g_p \equiv 0$; $\int_{\mathbf{0}}^{\infty} (\cdot) = \int_0^{\infty} \cdots \int_0^{\infty} (\cdot)$; *and* $\prod^{i(1,\ldots,p)} (\cdot)$ *is a special product symbol with the i-th factor to be located in the i-th position from left in the product*.

Proof. Since the Jacobian is $\mathrm{d}\mathbf{v} = (v_{11} \cdots v_{pp})^{(p-1)/2} \mathrm{d}\mathbf{u}$, the pdf of $\mathbf{u}$ is written as

$$w_p(\mathbf{u} \mid \boldsymbol{\rho}, n) = C_p \exp\{-\mathrm{tr}(\mathbf{P}^{-1}\mathbf{DRD}) / 2\} \mid \mathbf{DRD} \mid^{(n-p-1)/2} (v_{11} \cdots v_{pp})^{(p-1)/2}.$$

Then, the marginal pdf of $\mathbf{r}$ or $\mathbf{R}$ is given as follows.

$$f_p(\mathbf{r}\,|\,\boldsymbol{\rho}, n)$$

$$= C_p \int_0^{\infty} \cdots \int_0^{\infty} \exp\{-\mathrm{tr}(\mathbf{P}^{-1}\mathbf{DRD})/2\}\,|\,\mathbf{DRD}\,|^{(n-p-1)/2}\,(v_{11}\cdots v_{pp})^{(p-1)/2}\mathrm{d}v_{11}\cdots \mathrm{d}v_{pp}$$

$$= C_p\,|\,\mathbf{R}\,|^{(n-p-1)/2} \int_{\mathbf{0}}^{\infty} \exp\left(-\sum\nolimits_{i=1}^{p} \rho^{ii} v_{ii}/2\right)\exp\left(-\sum\nolimits_{i=1}^{p-1}\sum\nolimits_{j=i+1}^{p} \rho^{ij} r_{ij} v_{ii}^{1/2} v_{jj}^{1/2}\right)$$

$$\times (v_{11}\cdots v_{pp})^{(n-2)/2}\mathrm{d}v_{11}\cdots \mathrm{d}v_{pp}$$

$$= C_p\,|\,\mathbf{R}\,|^{(n-p-1)/2} \int_{\mathbf{0}}^{\infty} \exp(-\rho^{11} v_{11}/2)\exp\left(-v_{11}^{1/2}\sum\nolimits_{j=2}^{p} \rho^{1j} r_{1j} v_{jj}^{1/2}\right) v_{11}^{(n-2)/2}\mathrm{d}v_{11}$$

$$\times \exp(-\rho^{22} v_{22}/2)\exp\left(-v_{22}^{1/2}\sum\nolimits_{j=3}^{p} \rho^{2j} r_{2j} v_{jj}^{1/2}\right) v_{22}^{(n-2)/2}\mathrm{d}v_{22}$$

$$\vdots$$

$$\times \exp(-\rho^{p-1,p-1} v_{p-1,p-1}/2)\exp\left(-v_{p-1,p-1}^{1/2} \rho^{p-1,p} r_{p-1,p} v_{pp}^{1/2}\right) v_{p-1,p-1}^{(n-2)/2}\mathrm{d}v_{p-1,p-1}$$

$$\times \exp(-\rho^{pp} v_{pp}/2) v_{pp}^{(n-2)/2}\mathrm{d}v_{pp}$$

$$= C_p\,|\,\mathbf{R}\,|^{(n-p-1)/2} \int_{\mathbf{0}}^{\infty} \exp(-\rho^{11} v_{11}/2)\exp(-g_1 v_{11}^{1/2}) v_{11}^{(n-2)/2}\mathrm{d}v_{11}$$

$$\times \exp(-\rho^{22} v_{22}/2)\exp(-g_2 v_{22}^{1/2}) v_{22}^{(n-2)/2}\mathrm{d}v_{22}$$

$$\vdots$$

$$\times \exp(-\rho^{p-1,p-1} v_{p-1,p-1}/2)\exp(-g_{p-1} v_{p-1,p-1}^{1/2}) v_{p-1,p-1}^{(n-2)/2}\mathrm{d}v_{p-1,p-1}$$

$$\times \exp(-\rho^{pp} v_{pp}/2) v_{pp}^{(n-2)/2}\mathrm{d}v_{pp}$$

$$= C_p\,|\,\mathbf{R}\,|^{(n-p-1)/2} \int_{\mathbf{0}}^{\infty} \prod^{i(1,\ldots,p)} \exp\left(-\frac{\rho^{ii} v_{ii}}{2} - g_i v_{ii}^{1/2}\right) v_{ii}^{(n-2)/2}\mathrm{d}v_{ii}\,,$$

which gives the required result. Q.E.D.

Theorem S1. *When* $p = 2$ *to* 4, *the marginal joint pdf's of* $\mathbf{r}$ *under the elliptical distribution are*

(i) $p = 2$

$$f_2(r_{12}\,|\,\rho_{12}, n) = \frac{2^{n-2}(1-\rho_{12}^2)^{n/2}(1-r_{12}^2)^{(n-3)/2}}{\pi\,\Gamma(n-1)} \sum_{k_1=0}^{\infty} \Gamma\left(\frac{n+k_1}{2}\right)^2 \frac{(2\rho_{12} r_{12})^{k_1}}{k_1!},$$

where $\Gamma(\cdot)^2 = \{\Gamma(\cdot)\}^2$, *which is well-documented* (*see, e.g., Anderson*, 1958, 2003, *Theorem* 4.2.2; *Muirhead*, 1982, *Equation* (11), *Section* 5.1.3).

(ii) $p = 3$

$$f_3(r_{12}, r_{13}, r_{23} \mid \rho_{12}, \rho_{13}, \rho_{23}, n)$$

$$= \frac{(1+2r_{12}r_{13}r_{23} - r_{12}^2 - r_{13}^2 - r_{23}^2)^{(n-4)/2}}{2^{3n/2}(1+2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2)^{n/2}\Gamma_3(n/2)}$$

$$\times \sum_{k_1,k_2=0}^{\infty} \sum_{m_{11}=0}^{k_1} \frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)\Gamma\left(\frac{n+k_1+k_2-m_{11}}{2}\right)(-1)^{k_1+k_2}}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(\rho^{33}/2)^{(n+k_1+k_2-m_{11})/2}}$$

$$\times \frac{(\rho^{12}r_{12})^{m_{11}}(\rho^{13}r_{13})^{k_1-m_{11}}(\rho^{23}r_{23})^{k_2}}{m_{11}!(k_1-m_{11})!k_2!},$$

where $\sum_{k_1,k_2=0}^{\infty}(\cdot) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(\cdot)$.

(iii) $p = 4$

$$f_4(r_{12}, r_{13}, \ldots, r_{34} \mid \rho_{12}, \rho_{13}, \ldots, \rho_{34}, n)$$

$$= \frac{|\mathbf{R}|^{(n-5)/2}}{2^{2n}|\mathbf{P}|^{n/2}\Gamma_4(n/2)} \sum_{k_1,k_2,k_3=0}^{\infty} \sum_{m_{11}=0}^{k_1} \sum_{m_{21}=0}^{k_1-m_{11}} \sum_{m_{22}=0}^{k_2} (-1)^{k_1+k_2+k_3}$$

$$\times \frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)\Gamma\left(\frac{n+k_3+m_{21}+m_{22}}{2}\right) \times\Gamma\left(\frac{n+k_1+k_2+k_3-m_{11}-m_{21}-m_{22}}{2}\right)}{\left\{\begin{array}{l}(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(\rho^{33}/2)^{(n+k_3+m_{21}+m_{22})/2} \\ \times(\rho^{44}/2)^{(n+k_1+k_2+k_3-m_{11}-m_{21}-m_{22})/2}\end{array}\right\}}$$

$$\times \frac{(\rho^{12}r_{12})^{m_{11}}(\rho^{13}r_{13})^{m_{21}}(\rho^{14}r_{14})^{k_1-m_{11}-m_{21}}(\rho^{23}r_{23})^{m_{22}}(\rho^{24}r_{24})^{k_2-m_{22}}(\rho^{34}r_{34})^{k_3}}{m_{11}!m_{21}!(k_1-m_{11}-m_{21})!m_{22}!(k_2-m_{22})!k_3!}.$$

Proof. It is known that the pdf of **r** under normality holds under the elliptical distribution (for the bivariate case, see Ali & Joarder, 1991, Theorem; for the p-variate case, Joarder & Ali, 1992, Theorem 3.1). Then, the pdf's are derived under normality.

(i) $p = 2$: In Lemma S1, the first integral with respect to v_{11} is given as

$$\int_0^\infty \exp\left(-\frac{\rho^{11}v_{11}}{2}\right)\exp(-g_1 v_{11}^{1/2})v_{11}^{(n-2)/2}\mathrm{d}v_{11}$$

$$=\sum_{k_1=0}^{\infty}\int_0^\infty \exp\left(-\frac{\rho^{11}v_{11}}{2}\right)\frac{(-g_1)^{k_1}v_{11}^{(n+k_1-2)/2}}{k_1!}\mathrm{d}v_{11}$$

$$=\sum_{k_1=0}^{\infty}\frac{\Gamma\{(n+k_1)/2\}(-g_1)^{k_1}}{(\rho^{11}/2)^{(n+k_1)/2}k_1!}.$$

Using this result, the integral in Lemma S1 up to v_{22} becomes

$$\int_0^\infty \prod^{i(1,2)}\exp\left(-\frac{\rho^{ii}v_{ii}}{2}-g_i v_{ii}^{1/2}\right)v_{ii}^{(n-2)/2}\mathrm{d}v_{ii}$$

$$=\int_0^\infty \sum_{k_1=0}^{\infty}\frac{\Gamma\{(n+k_1)/2\}(-g_1)^{k_1}}{(\rho^{11}/2)^{(n+k_1)/2}k_1!}\exp\left(-\frac{\rho^{22}v_{22}}{2}-g_2 v_{22}^{1/2}\right)v_{22}^{(n-2)/2}\mathrm{d}v_{22}$$

$$=\int_0^\infty \sum_{k_1,k_2=0}^{\infty}\frac{\Gamma\{(n+k_1)/2\}(-\sum_{j=2}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1}}{(\rho^{11}/2)^{(n+k_1)/2}k_1!}\exp\left(-\frac{\rho^{22}v_{22}}{2}\right)\frac{(-g_2)^{k_2}v_{22}^{k_2/2}}{k_2!}v_{22}^{(n-2)/2}\mathrm{d}v_{22}$$

$$=\sum_{k_1,k_2=0}^{\infty}\sum_{m_{11}=0}^{k_1}\frac{\Gamma\{(n+k_1)/2\}}{(\rho^{11}/2)^{(n+k_1)/2}k_1!k_2!}(-g_2)^{k_2}$$

$$\times(-1)^{k_1}\binom{k_1}{m_{11}}(\sum_{j=3}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1-m_{11}}\int_0^\infty(\rho^{12}r_{12}v_{22}^{1/2})^{m_{11}}\exp\left(-\frac{\rho^{22}v_{22}}{2}\right)v_{22}^{(n+k_2-2)/2}\mathrm{d}v_{22}$$

$$=\sum_{k_1,k_2=0}^{\infty}\sum_{m_{11}=0}^{k_1}\frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)(-1)^{k_1+k_2}(\rho^{12}r_{12})^{m_{11}}(\sum_{j=3}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1-m_{11}}}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(k_1-m_{11})!k_2!m_{11}!}g_2^{k_2}.$$

In the above result, when $p = 2$, we have $(\sum_{j=3}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1-m_{11}}=0^{k_1-m_{11}}$ and $g_2^{k_2}=0^{k_2}$, which are defined to be 1 only when $k_1-m_{11}=0$ and $k_2=0$ otherwise 0 due to the properties of the binominal and exponential expansions, respectively. Consequently, when $p = 2$, the above integral becomes simplified:

$$\int_{\mathbf{0}}^{\infty}\prod^{i(1,2)}\exp\left(-\frac{\rho^{ii}v_{ii}}{2}-g_i v_{ii}^{1/2}\right)v_{ii}^{(n-2)/2}\mathrm{d}v_{ii}=\sum_{k_1=0}^{\infty}\frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_1}{2}\right)(-1)^{k_1}(\rho^{12}r_{12})^{k_1}}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_1)/2}k_1!}$$

$$=2^n(1-\rho_{12}^2)^n\sum_{k_1=0}^{\infty}\Gamma\left(\frac{n+k_1}{2}\right)^2\frac{(2\rho_{12}r_{12})^{k_1}}{k_1!}.$$

Using the above expression we obtain

$$f_2(r_{12}\mid\rho_{12},n)=C_2(1-r_{12}^2)^{(n-3)/2}\int_{\mathbf{0}}^{\infty}\prod^{i(1,2)}\exp\left(-\frac{\rho^{ii}v_{ii}}{2}-g_i v_{ii}^{1/2}\right)v_{ii}^{(n-2)/2}\mathrm{d}v_{ii}$$

$$=\frac{(1-r_{12}^2)^{(n-3)/2}}{2^n(1-\rho_{12}^2)^{n/2}\Gamma_2(n/2)}2^n(1-\rho_{12}^2)^n\sum_{k_1=0}^{\infty}\Gamma\left(\frac{n+k_1}{2}\right)^2\frac{(2\rho_{12}r_{12})^{k_1}}{k_1!}$$

$$=\frac{2^{n-2}(1-\rho_{12}^2)^{n/2}(1-r_{12}^2)^{(n-3)/2}}{\pi\Gamma(n-1)}\sum_{k_1=0}^{\infty}\Gamma\left(\frac{n+k_1}{2}\right)^2\frac{(2\rho_{12}r_{12})^{k_1}}{k_1!}.$$

In the above result, $\Gamma_2(n/2)=\pi^{1/2}\Gamma\{(n-1)/2\}\Gamma(n/2)=2^{-(n-2)}\pi\Gamma(n-1)$ is used with the Legendre duplication formula $\Gamma(2z)=2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\{z+(1/2)\}$ (see Erdélyi, 1953a, Section 1.2, Equation (15); DLMF, 2021, Equation 5.5.5, https://dlmf.nist.gov/5.5.E5).

(ii) p = 3: The integral in Lemma S1 up to v_{33} is obtained as

$$\int_{\mathbf{0}}^{\infty}\prod^{i(1,2,3)}\exp\left(-\frac{\rho^{ii}v_{ii}}{2}-g_i v_{ii}^{1/2}\right)v_{ii}^{(n-2)/2}\mathrm{d}v_{ii}$$

$$=\sum_{k_1,k_2=0}^{\infty}\sum_{m_{11}=0}^{k_1}\frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)(-1)^{k_1+k_2}(\rho^{12}r_{12})^{m_{11}}}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(k_1-m_{11})!k_2!m_{11}!}$$

$$\times\int_0^{\infty}g_2^{k_2}\left(\sum\nolimits_{j=3}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2}\right)^{k_1-m_{11}}\exp\left(-\frac{\rho^{33}v_{33}}{2}-g_3 v_{33}^{1/2}\right)v_{33}^{(n-2)/2}\mathrm{d}v_{33}$$

$$=\sum_{k_1,k_2,k_3=0}^{\infty}\sum_{m_{11}=0}^{k_1}\sum_{m_{21}=0}^{k_1-m_{11}}\sum_{m_{22}=0}^{k_2}\frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)\Gamma\left(\frac{n+k_3+m_{21}+m_{22}}{2}\right)(-1)^{k_1+k_2+k_3}}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(\rho^{33}/2)^{(n+k_3+m_{21}+m_{22})/2}}$$

$$\times\frac{(\rho^{12}r_{12})^{m_{11}}(\rho^{13}r_{13})^{m_{21}}(\rho^{23}r_{23})^{m_{22}}\left(\sum\nolimits_{j=4}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2}\right)^{k_1-m_{11}-m_{21}}\left(\sum\nolimits_{j=4}^{p}\rho^{2j}r_{2j}v_{jj}^{1/2}\right)^{k_2-m_{22}}}{(k_1-m_{11}-m_{21})!(k_2-m_{22})!m_{11}!m_{21}!m_{22}!k_3!}g_3^{k_3}.$$

As before, when $p = 3$, the above integral vanishes unless $k_1 - m_{11} - m_{21} = 0$, $k_2 - m_{22} = 0$ and $k_3 = 0$, which gives

$$\int_0^\infty \prod^{i(1,2,3)} \exp\left(-\frac{\rho^{ii} v_{ii}}{2} - g_i v_{ii}^{1/2}\right) v_{ii}^{(n-2)/2} \mathrm{d}v_{ii}$$

$$= \sum_{k_1,k_2=0}^{\infty} \sum_{m_{11}=0}^{k_1} \frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)\Gamma\left(\frac{n+k_1+k_2-m_{11}}{2}\right)(-1)^{k_1+k_2}}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(\rho^{33}/2)^{(n+k_1+k_2-m_{11})/2}} \times \frac{(\rho^{12} r_{12})^{m_{11}} (\rho^{13} r_{13})^{k_1-m_{11}} (\rho^{23} r_{23})^{k_2}}{m_{11}!(k_1-m_{11})!k_2!},$$

yielding the required result.

(iii) $p = 4$: The integral up to v_{44} is derived as

$$\int_0^\infty \prod^{i(1,\ldots,4)} \exp\left(-\frac{\rho^{ii} v_{ii}}{2} - g_i v_{ii}^{1/2}\right) v_{ii}^{(n-2)/2} \mathrm{d}v_{ii}$$

$$= \sum_{k_1,k_2,k_3=0}^{\infty} \sum_{m_{11}=0}^{k_1} \sum_{m_{21}=0}^{k_1-m_{11}} \sum_{m_{22}=0}^{k_2} \frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)\Gamma\left(\frac{n+k_3+m_{21}+m_{22}}{2}\right)(-1)^{k_1+k_2+k_3}}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(\rho^{33}/2)^{(n+k_3+m_{21}+m_{22})/2}}$$

$$\times \int_0^\infty \frac{(\rho^{12} r_{12})^{m_{11}} (\rho^{13} r_{13})^{m_{21}} (\rho^{23} r_{23})^{m_{22}} (\sum_{j=4}^{p} \rho^{1j} r_{1j} v_{jj}^{1/2})^{k_1-m_{11}-m_{21}} (\sum_{j=4}^{p} \rho^{2j} r_{2j} v_{jj}^{1/2})^{k_2-m_{22}}}{(k_1-m_{11}-m_{21})!(k_2-m_{22})!m_{11}!m_{21}!m_{22}!k_3!} g_3^{k_3}$$

$$\times \exp\left(-\frac{\rho^{44} v_{44}}{2} - g_4 v_{44}^{1/2}\right) v_{44}^{(n-2)/2} \mathrm{d}v_{44}$$

$$= \sum_{k_1,\ldots,k_4=0}^{\infty} \sum_{m_{11}=0}^{k_1} \sum_{m_{21}=0}^{k_1-m_{11}} \sum_{m_{22}=0}^{k_2} \sum_{m_{31}=0}^{k_1-m_{11}-m_{21}} \sum_{m_{32}=0}^{k_2-m_{22}} \sum_{m_{33}=0}^{k_3} (-1)^{k_1+\ldots+k_4}$$

$$\times \frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)\Gamma\left(\frac{n+k_3+m_{21}+m_{22}}{2}\right)\Gamma\left(\frac{n+k_4+m_{31}+m_{32}+m_{33}}{2}\right)}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(\rho^{33}/2)^{(n+k_3+m_{21}+m_{22})/2}(\rho^{44}/2)^{(n+k_4+m_{31}+m_{32}+m_{33})/2}}$$

$$\times (\rho^{12} r_{12})^{m_{11}} (\rho^{13} r_{13})^{m_{21}} (\rho^{23} r_{23})^{m_{22}} (\rho^{14} r_{14})^{m_{31}} (\rho^{24} r_{24})^{m_{32}} (\rho^{34} r_{34})^{m_{33}}$$

$$\times \frac{(\sum_{j=5}^{p} \rho^{1j} r_{1j} v_{jj}^{1/2})^{k_1-m_{11}-m_{21}-m_{31}} (\sum_{j=5}^{p} \rho^{2j} r_{2j} v_{jj}^{1/2})^{k_2-m_{22}-m_{32}} (\sum_{j=5}^{p} \rho^{3j} r_{3j} v_{jj}^{1/2})^{k_3-m_{33}}}{(k_1-m_{11}-m_{21}-m_{31})!(k_2-m_{22}-m_{32})!(k_3-m_{33})!m_{11}!m_{21}!\cdots m_{33}!k_4!} g_4^{k_4}.$$

As before when $p = 4$, the above integral vanishes unless $k_1 - m_{11} - m_{21} - m_{31} = 0$, $k_2 - m_{22} - m_{32} = 0$, $k_3 - m_{33} = 0$ and $k_4 = 0$, which gives

$$\int_0^{\infty} \prod^{i(1,\ldots,4)} \exp\left(-\frac{\rho^{ii} v_{ii}}{2} - g_i v_{ii}^{1/2} \right) v_{ii}^{(n-2)/2} \mathrm{d}v_{ii}$$

$$= \sum_{k_1,k_2,k_3=0}^{\infty} \sum_{m_{11}=0}^{k_1} \sum_{m_{21}=0}^{k_1-m_{11}} \sum_{m_{22}=0}^{k_2} (-1)^{k_1+k_2+k_3}$$

$$\times \frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)\Gamma\left(\frac{n+k_3+m_{21}+m_{22}}{2}\right) \times\Gamma\left(\frac{n+k_1+k_2+k_3-m_{11}-m_{21}-m_{22}}{2}\right)}{\left\{\begin{array}{l}(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(\rho^{33}/2)^{(n+k_3+m_{21}+m_{22})/2} \\ \times(\rho^{44}/2)^{(n+k_1+k_2+k_3-m_{11}-m_{21}-m_{22})/2}\end{array}\right\}}$$

$$\times \frac{(\rho^{12}r_{12})^{m_{11}}(\rho^{13}r_{13})^{m_{21}}(\rho^{14}r_{14})^{k_1-m_{11}-m_{21}}(\rho^{23}r_{23})^{m_{22}}(\rho^{24}r_{24})^{k_2-m_{22}}(\rho^{34}r_{34})^{k_3}}{m_{11}!m_{21}!(k_1-m_{11}-m_{21})!m_{22}!(k_2-m_{22})!k_3!}.$$

Then, the required result follows. Q.E.D.

Theorem S2. *The marginal joint pdf of* **r** *under the p-variate elliptical distribution is*

$$f_p(\mathbf{r}\,|\,\boldsymbol{\rho},n)$$

$$= C_p\,|\,\mathbf{R}\,|^{(n-p-1)/2}$$

$$\times \sum_{k_1,\ldots,k_{p-1}=0}^{\infty} \sum_{m_{11}=0}^{k_1} \sum_{m_{21}=0}^{k_1-m_{11}} \cdots \sum_{m_{p-2,1}=0}^{k_1-m_{11}\cdots-m_{p-3,1}} \sum_{m_{22}=0}^{k_2} \cdots \sum_{m_{p-2,2}=0}^{k_2-m_{22}\cdots-m_{p-3,2}} \cdots \sum_{m_{p-2.p-2}=0}^{k_{p-2}} (-1)^{k_1+\ldots+k_{p-1}}$$

$$\times\left[\prod_{i=1}^{p-1}\Gamma\left(\frac{n+k_i+m_{i-1,1}+\ldots+m_{i-1,i-1}}{2}\right)/(\rho^{ii}/2)^{(n+k_i+m_{i-1,1}+\ldots+m_{i-1,i-1})/2}\right]$$

$$\times\Gamma\left(\frac{n+k_1+\ldots+k_{p-1}-m_{11}-\ldots.-m_{p-2,1}\ldots-m_{p-2,p-2}}{2}\right)$$

$$\times 1/(\rho^{pp}/2)^{(n+k_1+\ldots+k_{i-1}-m_{11}-\ldots-m_{p-2,1}-\ldots-m_{p-2,p-2})/2}$$

$$\times\prod_{i=1}^{p-1}\left\{\prod_{j=i+1}^{p-1}\frac{(\rho^{ij}r_{ij})^{m_{j-1,i}}}{m_{j-1,i}!}\right\}\frac{(\rho^{ip}r_{ip})^{k_i-m_{ii}-\ldots-m_{p-2,i}}}{(k_i-m_{ii}-\ldots-m_{p-2,i})!},$$

where $m_{0j}=0\ \ (j=0,1)$.

Proof. The derivation is shown by induction. The results of the integral up to v_{44} are given in Theorem S1. Assume that the integral up to v_{qq} in Lemma S1 holds as shown below. Then, the result up to $v_{q+1,q+1}$ is given as follows:

$$\sum_{k_1,...,k_q=0}^{\infty}\sum_{m_{11}=0}^{k_1}\sum_{m_{21}=0}^{k_1-m_{11}}\sum_{m_{22}=0}^{k_2}\cdots\sum_{m_{q-1,1}=0}^{k_1-m_{11}\cdots-m_{q-2,1}}\sum_{m_{q-1,2}=0}^{k_2-m_{22}\cdots-m_{q-2,2}}\cdots\sum_{m_{q-1.q-1}=0}^{k_{q-1}}(-1)^{k_1+...+k_q}$$
$$\times\left[\prod_{i=1}^{q}\Gamma\left(\frac{n+k_i+m_{i-1,1}+...+m_{i-1,i-1}}{2}\right)/(\rho^{ii}/2)^{(n+k_i+m_{i-1,1}+...+m_{i-1,i-1})/2}\right]$$
$$\times(\rho^{12}r_{12})^{m_{11}}(\rho^{13}r_{13})^{m_{21}}(\rho^{23}r_{23})^{m_{22}}\cdots(\rho^{1q}r_{1q})^{m_{q-1,1}}\cdots(\rho^{q-1,q}r_{q-1,q})^{m_{q-1,q-1}}$$
$$\times\int_0^{\infty}\frac{(\sum_{j=q+1}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1-m_{11}\cdots-m_{q-1,1}}\cdots(\sum_{j=q+1}^{p}\rho^{q-1,j}r_{q-1,j}v_{jj}^{1/2})^{k_{q-1}-m_{q-1.q-1}}}{(k_1-m_{11}\cdots-m_{q-1,1})!\cdots(k_{q-1}-m_{q-1.q-1})!m_{11}!m_{21}!\cdots m_{q-1,q-1}!k_q!}g_q^{k_q}$$
$$\times\exp\left(-\frac{\rho^{q+1,q+1}v_{q+1,q+1}}{2}-g_{q+1}v_{q+1,q+1}^{1/2}\right)v_{q+1,q+1}^{(n-2)/2}\mathrm{d}v_{q+1,q+1}.$$

The numerator of the above integrand is expanded as

$$(\sum_{j=q+1}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1-m_{11}-,,,-m_{q-1,1}}\cdots(\sum_{j=q+1}^{p}\rho^{q-1,j}r_{q-1,j}v_{jj}^{1/2})^{k_{q-1}-m_{q-1.q-1}}$$
$$=\sum_{m_{q1}=0}^{k_1-m_{11}-...-m_{q-1,1}}\binom{k_1-m_{11}-...-m_{q-1,1}}{m_{q1}}(\rho^{1,q+1}r_{1,q+1})^{m_{q1}}v_{q+1,q+1}^{m_{q1}/2}(\sum_{j=q+2}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1-m_{11}-...-m_{q1}}$$
$$\cdots\sum_{m_{q,q-1}=0}^{k_{q-1}}\binom{k_{q-1}-m_{q-1.q-1}}{m_{q,q-1}}(\rho^{q-1,q+1}r_{q-1,q+1})^{m_{q,q-1}}v_{q+1,q+1}^{m_{q,q-1}/2}(\sum_{j=q+2}^{p}\rho^{q-1,j}r_{q-1,j}v_{jj}^{1/2})^{k_{q-1}-m_{q-1.q-1}-m_{q,q-1}}.$$

The factor $g_q^{k_q}$ in the integrand becomes

$$g_q^{k_q}=(\sum_{j=q+1}^{p}\rho^{qj}r_{qj}v_{jj}^{1/2})^{k_q}$$
$$=\sum_{m_{qq}=0}^{k_q}\binom{k_q}{m_{qq}}(\rho^{q,q+1}r_{q,q+1})^{m_{qq}}v_{q+1,q+1}^{m_{qq}/2}(\sum_{j=q+2}^{p}\rho^{qj}r_{qj}v_{jj}^{1/2})^{k_q-m_{qq}}.$$

The remaining factor in the integrand is also expanded:

$$\exp\left(-\frac{\rho^{q+1,q+1}v_{q+1,q+1}}{2}-g_{q+1}v_{q+1,q+1}^{1/2}\right)v_{q+1,q+1}^{(n-2)/2}$$
$$=\exp\left(-\frac{\rho^{q+1,q+1}v_{q+1,q+1}}{2}\right)\sum_{k_{q+1}=0}^{\infty}\frac{(-1)^{k_{q+1}}g_{q+1}^{k_{q+1}}v_{q+1,q+1}^{(n+k_{q+1}-2)/2}}{k_{q+1}!}.$$

Then, the integral becomes

$$\int_0^\infty \frac{(\sum_{j=q+1}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1-m_{11}-\ldots-m_{q-1,1}}\cdots(\sum_{j=q+1}^{p}\rho^{q-1,j}r_{q-1,j}v_{jj}^{1/2})^{k_{q-1}}}{(k_1-m_{11}-\ldots-m_{q-1,1})!\cdots(k_{q-1}-m_{q-1.q-1})!m_{11}!m_{21}!\cdots m_{q-1,q-1}!k_q!}g_q^{k_q}$$

$$\times\exp\left(-\frac{\rho^{q+1,q+1}v_{q+1,q+1}}{2}-g_{q+1}v_{q+1,q+1}^{1/2}\right)v_{q+1,q+1}^{(n-2)/2}\mathrm{d}v_{q+1,q+1}$$

$$=\sum_{k_{q+1}=0}^{\infty}\sum_{m_{q1}=0}^{k_1-m_{11}\cdots-m_{q-1,1}}\cdots\sum_{m_{q,q-1}=0}^{k_{q-1}}\sum_{m_{qq}=0}^{k_q}(-1)^{k_{q+1}}\binom{k_1-m_{11}-\ldots-m_{q-1,1}}{m_{q1}}\cdots\binom{k_{q-1}-m_{q-1,q-1}}{m_{q,q-1}}\binom{k_q}{m_{qq}}$$

$$\times\frac{(\rho^{1,q+1}r_{1,q+1})^{m_{q1}}\cdots(\rho^{q-1,q+1}r_{q-1,q+1})^{m_{q,q-1}}(\rho^{q,q+1}r_{q,q+1})^{m_{qq}}}{(k_1-m_{11}-\ldots-m_{q-1,1})!\cdots(k_{q-1}-m_{q-1.q-1})!m_{11}!m_{21}!\cdots m_{q-1,q-1}!k_q!}$$

$$\times\int_0^\infty\exp\left(-\frac{\rho^{q+1,q+1}v_{q+1,q+1}}{2}\right)v_{q+1,q+1}^{(n+k_{q+1}+m_{q1}+\cdots+m_{qq}-2)/2}\mathrm{d}v_{q+1,q+1}$$

$$\times(\sum_{j=q+2}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1-m_{11}-\ldots-m_{q1}}$$

$$\cdots(\sum_{j=q+2}^{p}\rho^{q-1,j}r_{q-1,j}v_{jj}^{1/2})^{k_{q-1}-m_{q-1,q-1}-m_{q,q-1}}(\sum_{j=q+2}^{p}\rho^{qj}r_{qj}v_{jj}^{1/2})^{k_q-m_{qq}}$$

$$\times g_{q+1}^{k_{q+1}}.$$

Note that in the above result, the integral becomes

$$\int_0^\infty\exp\left(-\frac{\rho^{q+1,q+1}v_{q+1,q+1}}{2}\right)v_{q+1,q+1}^{(n+k_{q+1}+m_{q1}+\ldots+m_{qq}-2)/2}\mathrm{d}v_{q+1,q+1}$$

$$=\Gamma\left(\frac{n+k_{q+1}+m_{q1}+\ldots+m_{qq}}{2}\right)/(\rho^{q+1,q+1}/2)^{(n+k_{q+1}+m_{q1}+\ldots+m_{qq})/2}$$

and when $q+1=p$, the whole above result vanishes unless $k_1-m_{11}\ldots-m_{q1}=0,\cdots,\ k_{q-1}-m_{q-1,q-1}-m_{q,q-1}=0$, $k_q-m_{qq}=0$ and $k_{q+1}=0$. Among the expanded factorials for

$$\binom{k_1-m_{11}-\ldots-m_{q-1,1}}{m_{q1}}\cdots\binom{k_{q-1}-m_{q-1,q-1}}{m_{q,q-1}}\binom{k_q}{m_{qq}},$$

those of the numerator are canceled and the half set of the factorials in the denominator become 0! = 1 due to $k_1-m_{11}-\ldots-m_{q1}=0,\ldots,$ $k_{q-1}-m_{q-1,q-1}-m_{q,q-1}=0$ and $k_q-m_{qq}=0$ leaving only $m_{q1}!\cdots m_{q,q-1}!m_{qq}!$. Then, we have

$$
\begin{aligned}
& f_p(\mathbf{r}\,|\,\boldsymbol{\rho},n) \\
&= C_p\,|\,\mathbf{R}\,|^{(n-p-1)/2} \\
&\times \sum_{k_1,\dots,k_p=0}^{\infty}\sum_{m_{11}=0}^{k_1}\sum_{m_{21}=0}^{k_1-m_{11}}\sum_{m_{22}=0}^{k_2}\cdots\sum_{m_{p-1,1}=0}^{k_1-m_{11}-\dots-m_{p-2,1}}\sum_{m_{p-1,2}=0}^{k_2-m_{22}-\dots-m_{p-2,2}}\cdots\sum_{m_{p-1.p-1}=0}^{k_{p-1}}(-1)^{k_1+\dots+k_p} \\
&\times\left[\prod\nolimits_{i=1}^{p}\Gamma\left(\frac{n+k_i+m_{i-1,1}+\dots+m_{i-1,i-1}}{2}\right)/(\rho^{ii}/2)^{(n+k_i+m_{i-1,1}+\dots+m_{i-1,i-1})/2}\right] \\
&\times\frac{(\rho^{12}r_{12})^{m_{11}}(\rho^{13}r_{13})^{m_{21}}(\rho^{23}r_{23})^{m_{22}}\cdots(\rho^{1p}r_{1p})^{m_{p-1,1}}\cdots(\rho^{p-1,p}r_{p-1.p})^{k_{p-1}}}{m_{11}!m_{21}!\cdots m_{p-1,p-2}!k_{p-1}!} \\
&= C_p\,|\,\mathbf{R}\,|^{(n-p-1)/2} \\
&\times \sum_{k_1,\dots,k_{p-1}=0}^{\infty}\sum_{m_{11}=0}^{k_1}\sum_{m_{21}=0}^{k_1-m_{11}}\cdots\sum_{m_{p-2,1}=0}^{k_1-m_{11}-\dots-m_{p-3,1}}\sum_{m_{22}=0}^{k_2}\cdots\sum_{m_{p-2,2}=0}^{k_2-m_{22}-\dots-m_{p-3,2}}\cdots\sum_{m_{p-2.p-2}=0}^{k_{p-2}}(-1)^{k_1+\dots+k_{p-1}} \\
&\times\frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)\cdots\Gamma\left(\frac{n+k_1+\dots+k_{p-1}-m_{11}-\dots-m_{p-2,1}-\dots-m_{p-2,p-2}}{2}\right)}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}\cdots(\rho^{pp}/2)^{(n+k_1+\dots+k_{p-1}-m_{11}-\dots-m_{p-2,1}-\dots-m_{p-2,p-2})/2}} \\
&\times\frac{(\rho^{12}r_{12})^{m_{11}}\cdots(\rho^{1p}r_{1p})^{k_1-m_{11}-\dots-m_{p-2,1}}(\rho^{23}r_{23})^{m_{22}}\cdots(\rho^{2p}r_{24})^{k_2-m_{22}-\dots-m_{p-2,2}}\cdots(\rho^{p-1,p}r_{p-1,p})^{k_{p-1}}}{m_{11}!\cdots(k_1-m_{11}-\dots-m_{p-2,1})!m_{22}!\cdots(k_2-m_{22}-\dots-m_{p-2,2})!\cdots k_{p-1}!},
\end{aligned}
$$

which gives the required result. Q.E.D.

The results of Theorem S1 can be seen as corollaries of Theorem S2. However, to have the relationship to the known result in the bivariate case and the initial condition(s) of the induction and to illustrate the sequential structure explicitly in the cases with small p's of practical importance, they are given as a theorem prior to Theorem S2 for the general p-variate cases.

S.2 Remarks on Theorems S1 and S2

The results of Theorems S1 and S2 show that they are given by $(p-1)$-fold infinite series along with their associated $\{(p-1)(p-2)/2\}$-fold nested series, which soon become excessively complicated when p becomes large. Joarder and Ali (1992, pp. 1958-1962) gave the results corresponding to those in Theorem S2 using different methods and expressions. Their results use a single infinite series with nested series and looks simpler than those in Theorem S2. However, two sets of expressions are equivalent or comparable in that our expressions can be summarized in a single infinite series when necessary. For instance, when p = 3, define $k_{12}\equiv k_1+k_2$. Then we have

$$\sum_{k_1,k_2=0}^{\infty}\sum_{m_{11}=0}^{k_1} t(k_1,k_2,k_1+k_2,m_{11}) = \sum_{k_{12}=0}^{\infty}\sum_{k_1=0}^{k_{12}}\sum_{m_{11}=0}^{k_1} t(k_1,k_{12}-k_1,k_{12},m_{11}),$$

where $t(k_1,k_2,k_{12},m_{11})$ is a function of the four arguments when $k_{12}=k_1+k_2$ is temporarily seen as an independent variable.

Similar series expansions are available by change of variables in integration. Recall that the integral expression in Lemma S1 with respect to v_{ii} is also given by $v_i \equiv v_{ii}^{1/2}\ (i=1,\dots,p)$:

$$
\begin{aligned}
&f_p(\mathbf{r}\,|\,\boldsymbol{\rho},n) \\
&= C_p\,|\,\mathbf{R}\,|^{(n-p-1)/2}\int_{\mathbf{0}}^{\infty}\prod^{i(1,\dots,p)}\exp\left(-\frac{\rho^{ii}v_{ii}}{2}-g_i v_{ii}^{1/2}\right)v_{ii}^{(n-2)/2}\mathrm{d}v_{ii} \\
&= C_p\,|\,\mathbf{R}\,|^{(n-p-1)/2}\int_{\mathbf{0}}^{\infty}\prod^{i(1,\dots,p)}2\exp\left(-\frac{\rho^{ii}v_i^2}{2}-g_i v_i\right)v_i^{n-1}\mathrm{d}v_i,
\end{aligned}
$$

where the first integral with respect to v_1 becomes

$$\int_0^{\infty}\exp\left(-\frac{\rho^{11}}{2}v_1^2-g_1v_1\right)v_1^{n-1}\mathrm{d}v_1=(\rho^{11})^{-n/2}\Gamma(n)e^{g_1^2/(4\rho^{11})}D_{-n}(g_1/\sqrt{\rho^{11}})$$

(Erdélyi, 1954, Section 6.3, Equation (13); Zwillinger, 2015, Section 3.462, Equation 1), where $D_{-n}(\cdot)$ is the parabolic cylinder function using the traditional Whittaker notation (Erdélyi, 1953b, Chapter 8; Magnus, Oberhettinger & Soni, 1966, Chapter VIII; Zwillinger, 2015, Sections 9.24-9.25; DLMF, 2021, Chapter 12), whose series expression is given by

$$
\begin{aligned}
&D_\nu(z) \\
&= 2^{\nu/2}e^{-z^2/4}\left\{\frac{\sqrt{\pi}}{\Gamma\{(1-\nu)/2\}}\,{}_1F_1\left(-\frac{\nu}{2};\ \frac{1}{2};\ \frac{z^2}{2}\right)-\frac{\sqrt{2\pi}\,z}{\Gamma(-\nu/2)}\,{}_1F_1\left(\frac{1-\nu}{2};\ \frac{3}{2};\ \frac{z^2}{2}\right)\right\},
\end{aligned}
$$

where

$${}_1F_1(a;b;z)=\sum_{k=0}^{\infty}\frac{(a)_k z^k}{(b)_k k!}$$

is the Kummer confluent hypergeometric function (Winkelbauer, 2014, Equation (6); Zwillinger, 2015, Section 9.210, Equation 1; DLMF, 2021, Chapter 13); and $(a)_k=a(a+1)\cdots(a+k-1)$ is the rising or ascending factorial using the Pochhammer symbol when k is a non-negative integer with $(a)_k=\Gamma(a+k)/\Gamma(a)\ (a>0,k\geq 0)$.

Using the series expression, we obtain

$$\int_0^\infty \exp\left(-\frac{\rho^{11}}{2}v_1^2 - g_1 v_1\right) v_1^{n-1} \mathrm{d}v_1$$
$$= (\rho^{11})^{-n/2}\Gamma(n)e^{g_1^2/(4\rho^{11})}D_{-n}(g_1/\sqrt{\rho^{11}})$$
$$= (2\rho^{11})^{-n/2}\Gamma(n)$$
$$\times\left\{\frac{\sqrt{\pi}}{\Gamma\{(n+1)/2\}}\,{}_1F_1\left(\frac{n}{2};\ \frac{1}{2};\ \frac{g_1^2}{2\rho^{11}}\right) - \frac{\sqrt{2\pi}g_1/\sqrt{\rho^{11}}}{\Gamma(n/2)}\,{}_1F_1\left(\frac{n+1}{2};\ \frac{3}{2};\ \frac{g_1^2}{2\rho^{11}}\right)\right\}.$$

It is intriguing to use this expression since $D_{-n}(\cdot)$ and ${}_1F_1(\cdot)$ are known functions with positive variable $g_1^2/(2\rho^{11})$ in ${}_1F_1(\cdot)$, expecting stable convergence. However, $g_1 = \sum_{j=2}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2} = \sum_{j=2}^{p}\rho^{1j}r_{1j}v_j$ is a function of $v_i\ (i=2,\dots,p)$, which will be variables for integration in the subsequent stages. Consequently, term-by-term integration in ${}_1F_1(\cdot)$ will again be required unless some simplification is derived.

The positive variable used in ${}_1F_1(\cdot)$ for the integral with respect to v_i 's can similarly be employed using v_{ii} 's, when desired, by considering even and odd terms as

$$\int_0^\infty \exp(-\rho^{11}v_{11}/2)\exp(-g_1 v_{11}^{1/2})v_{11}^{(n-2)/2}\mathrm{d}v_{11}$$
$$= \sum_{k_1=0}^{\infty}\int_0^\infty \exp(-\rho^{11}v_{11}/2)\frac{(-g_1)^{k_1}v_{11}^{(n+k_1-2)/2}}{k_1!}\mathrm{d}v_{11}$$
$$= \sum_{k_1=0}^{\infty}\int_0^\infty \exp(-\rho^{11}v_{11}/2)\left\{\frac{g_1^{2k_1}v_{11}^{(n+2k_1-2)/2}}{(2k_1)!} - \frac{g_1^{2k_1+1}v_{11}^{(n+2k_1-1)/2}}{(2k_1+1)!}\right\}\mathrm{d}v_{11}$$
$$= \sum_{k_1=0}^{\infty}\left\{\frac{\Gamma\{(n+2k_1)/2\}g_1^{2k_1}}{(\rho^{11}/2)^{(n+2k_1)/2}(2k_1)!} - \frac{\Gamma\{(n+2k_1+1)/2\}g_1^{2k_1+1}}{(\rho^{11}/2)^{(n+2k_1+1)/2}(2k_1+1)!}\right\}$$

$$= \left(\frac{2}{\rho^{11}}\right)^{n/2} \sum_{k_1=0}^{\infty} \left\{ \frac{\sqrt{\pi}\Gamma\{(n+2k_1)/2\}(g_1/\sqrt{2\rho^{11}})^{2k_1}}{\Gamma\{(2k_1+1)/2\}k_1!} - \frac{\sqrt{\pi}\Gamma\{(n+2k_1+1)/2\}(g_1/\sqrt{2\rho^{11}})^{2k_1+1}}{\Gamma\{(2k_1+3)/2\}k_1!} \right\}$$

$$= \left(\frac{2}{\rho^{11}}\right)^{n/2} \sqrt{\pi} \sum_{k_1=0}^{\infty} \Big[\{(2k_1+1)/2\}_{(n-1)/2}(g_1/\sqrt{2\rho^{11}})^{2k_1} - \{(2k_1+3)/2\}_{(n-2)/2}(g_1/\sqrt{2\rho^{11}})^{2k_1+1} \Big] / k_1!.$$

Closed form formulas for the product moments of $\mathbf{r}$ or raw moments of $|\mathbf{R}|$ (the scatter coefficient) are not given as applications of the main results, which are open problems. The integrals in Theorem S2 are given sequentially in element-wise methods yielding multiple series. Recently, the Holonomic gradient method to have the Fisher-Bingham integral was developed (see Nakayama et al., 2011), which is seen as a multivariate counterpart of the scaled parabolic cylinder function using an integral expression. This method has been applied to the distributions of statistics associated with the Wishart (Hashiguchi et al., 2013; Shimizu & Hashiguchi, 2019). Mura et al. (2019) showed the Holonomic property of the distribution of the sample correlation coefficient in the bivariate Wishart based on the formula of Hotelling (1953, Equation (25)) using the Gauss hypergeometric series (Abramowitz & Stegun, 1964/2014, Section 15.1; Zwillinger, 2015, Sections 9.1; Hankin, 2016; DLMF, 2021, Section 15.2 (i)). This finding suggests a similar property in the p-variate case and an efficient method to have the infinite series with a matrix argument (see also Pham-Gia & Choulakian, 2014; Pham-Gia & Thanh, 2016).

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