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Supplement I to the paper "Asymptotic cumulants of ability estimators using fallible item parameters" − Proofs, partial derivatives and tables (Corrected version) March, 2014

Haruhiko Ogasawara

This supplement includes Subsections A.2 to A.5 of the appendix of Ogasawara (2013) and additional Tables A1 to A4.

A.2 Expansions of $\hat{\boldsymbol{\gamma}}_{\theta_0}^{(k)}$ and $\hat{\mathbf{l}}_{\theta_0}^{(k)}\ (k=1,2,3)$

(a) $\hat{\boldsymbol{\gamma}}_{\theta_0}^{(k)}(k=1,2,3)$

By the Taylor series expansion and (2.6), it follows that

$$\hat{\boldsymbol{\gamma}}_{\theta_0}^{(k)} = \boldsymbol{\gamma}_{\theta_0}^{(k)} + \frac{\partial \boldsymbol{\gamma}_{\theta_0}^{(k)}}{\partial \boldsymbol{\alpha}_0{}'}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0) + \frac{1}{2}\frac{\partial^2 \boldsymbol{\gamma}_{\theta_0}^{(k)}}{(\partial \boldsymbol{\alpha}_0{}')^{<2>}}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0)^{<2>} + O_p(N^{-3/2})$$

$$= \boldsymbol{\gamma}_{\theta_0}^{(k)} + \frac{\partial \boldsymbol{\gamma}_{\theta_0}^{(k)}}{\partial \boldsymbol{\alpha}_0{}'}\left(\sum_{k=1}^{2}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(k)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(k)} - N^{-1}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\right) + \frac{1}{2}\frac{\partial^2 \boldsymbol{\gamma}_{\theta_0}^{(k)}}{(\partial \boldsymbol{\alpha}_0{}')^{<2>}}(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>}$$

$$+ O_p(N^{-3/2})$$

$$= (\boldsymbol{\gamma}_{\theta_0}^{(k)})_{O(1)} + \left(\frac{\partial \boldsymbol{\gamma}_{\theta_0}^{(k)}}{\partial \boldsymbol{\alpha}_0{}'}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)}\right)_{O_p(N^{-1/2})} + \left\{\frac{\partial \boldsymbol{\gamma}_{\theta_0}^{(k)}}{\partial \boldsymbol{\alpha}_0{}'}(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}) + \frac{1}{2}\frac{\partial^2 \boldsymbol{\gamma}_{\theta_0}^{(k)}}{(\partial \boldsymbol{\alpha}_0{}')^{<2>}}(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>}\right\}_{O_p(N^{-1})} + O_p(N^{-3/2})$$

$$\equiv (\boldsymbol{\gamma}_{\theta_0}^{(k)})_{O(1)} + (\boldsymbol{\gamma}_{\theta_0}^{(\Delta k)})_{O_p(N^{-1/2})} + (\boldsymbol{\gamma}_{\theta_0}^{(\Delta\Delta k)})_{O_p(N^{-1})} + O_p(N^{-3/2})\ \ (k=1,2,3).$$

(b) $\hat{\mathbf{l}}_{\theta_0}^{(1)} = \hat{l}_{\theta_0}^{(1)} = \dfrac{\partial \hat{\bar{l}}_{\theta_0}}{\partial \theta_0}$

$$= \left(\frac{\partial \overline{l}_{\theta_0}}{\partial \theta_0} \right)_{O_p(n^{-1/2})} + \left\{ \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0 '} \right)_{O_p(1)} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})} \right\}_{O_p(N^{-1/2})}$$

$$+ \Bigg[\left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0 '} \right)_{O_p(1)} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1})}$$

$$+ \left(\frac{1}{2} \frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0 ')^{<2>}} \right)_{O_p(1)} \{ (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>} \}_{O_p(N^{-1})} \Bigg]_{O_p(N^{-1})}$$

$$+ \Bigg[\left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0 '} \right)_{O_p(1)} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(3)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(3)})_{O_p(N^{-3/2})} + \left(\frac{1}{2} \frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0 ')^{<2>}} \right)_{O_p(1)}$$

$$\times \sum_{\otimes}^{2} \{ (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)}) \otimes (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0}) \}_{O_p(N^{-3/2})}$$

$$+ \left(\frac{1}{6} \frac{\partial^4 \overline{l}_{\theta_0}}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0 ')^{<3>}} \right)_{O_p(1)} \{ (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<3>} \}_{O_p(N^{-3/2})} \Bigg]_{O_p(N^{-3/2})} + O_p(N^{-2})$$

$$= \left(\frac{\partial \overline{l}_{\theta_0}}{\partial \theta_0} \right)_{O_p(n^{-1/2})} + \left\{ \left(\mathrm{E}_{\mathrm{T}\theta_0} \frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0 '} \right)_{O(1)} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})} \right\}_{O_p(N^{-1/2})}$$

$$+ \left\{ \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0 '} - \mathrm{E}_{\mathrm{T}\theta_0}(\cdot) \right)_{O_p(n^{-1/2})} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})} \right\}_{O_p(n^{-1/2} N^{-1/2})}$$

$$+ \Bigg[\left(\mathrm{E}_{\mathrm{T}\theta_0} \frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0 '} \right)_{O(1)} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1})}$$

$$+\left(\frac{1}{2}\mathrm{E}_{\mathrm{T}\theta_0}\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)_{O(1)}\{(\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})^{<2>}\}_{O_p(N^{-1})}\Bigg]_{O_p(N^{-1})}$$

$$+\Bigg[\left(\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}-\mathrm{E}_{\mathrm{T}\theta_0}(\cdot)\right)_{O_p(n^{-1/2})}(\boldsymbol{\Gamma}^{(2)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(2)}_{\boldsymbol{\alpha}_0}-N^{-1}\boldsymbol{\Lambda}^{-1}_{\boldsymbol{\alpha}_0}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1})}$$

$$+\frac{1}{2}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}-\mathrm{E}_{\mathrm{T}\theta_0}(\cdot)\right)_{O_p(n^{-1/2})}\{(\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})^{<2>}\}_{O_p(N^{-1})}\Bigg]_{O_p(n^{-1/2}N^{-1})}$$

$$+\Bigg[\left(\mathrm{E}_{\mathrm{T}\theta_0}\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}\right)_{O(1)}(\boldsymbol{\Gamma}^{(3)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(3)}_{\boldsymbol{\alpha}_0})_{O_p(N^{-3/2})}$$

$$+\frac{1}{2}\mathrm{E}_{\mathrm{T}\theta_0}\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<2>}}\sum_{\otimes}^{2}\{(\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})\otimes(\boldsymbol{\Gamma}^{(2)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(2)}_{\boldsymbol{\alpha}_0}-N^{-1}\boldsymbol{\Lambda}^{-1}_{\boldsymbol{\alpha}_0}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0})\}_{O_p(N^{-3/2})}$$

$$+\left(\frac{1}{6}\mathrm{E}_{\mathrm{T}_{\theta_0}}\frac{\partial^4\overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0')^{<3>}}\right)_{O(1)}\{(\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0}\mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})^{<3>}\}_{O_p(N^{-3/2})}\Bigg]_{O_p(N^{-3/2})}$$

$$+O_p(n^{-1/2}N^{-3/2})+O_p(N^{-2})$$

$$\equiv(l^{(1)}_{\theta_0})_{O_p(n^{-1/2})}+(l^{(\Delta 1)}_{\theta_0})_{O_p(N^{-1/2})}+(l^{(\Delta\Delta a1)}_{\theta_0})_{O_p(n^{-1/2}N^{-1/2})}+(l^{(\Delta\Delta b1)}_{\theta_0})_{O_p(N^{-1})}$$

$$+(l^{(\Delta\Delta\Delta a1)}_{\theta_0})_{O_p(n^{-1/2}N^{-1})}+(l^{(\Delta\Delta\Delta b1)}_{\theta_0})_{O_p(N^{-3/2})}+O_p(n^{-1/2}N^{-3/2})+O_p(N^{-2}),$$

where $\sum_{\otimes}^{2}\mathbf{x}\otimes\mathbf{y}=\mathbf{x}\otimes\mathbf{y}+\mathbf{y}\otimes\mathbf{x}$ and $\mathbf{A}-\mathrm{E}_{\mathrm{T}_{\theta_0}}(\cdot)=\mathbf{A}-\mathrm{E}_{\mathrm{T}_{\theta_0}}(\mathbf{A})$.

(c) $$\hat{\mathbf{l}}^{(2)}_{\theta_0}=\left\{\hat{m}\frac{\partial\hat{\overline{l}}_{\theta_0}}{\partial\theta_0},\ \left(\frac{\partial\hat{\overline{l}}_{\theta_0}}{\partial\theta_0}\right)^2\right\}'$$

In the above expression,

$$\hat{m} \equiv \frac{\partial^2 \hat{\overline{l}}_{\theta_0}}{\partial \theta_0^2} - \hat{\lambda}_{\theta_0} \quad \left(\text{recall } \lambda_{\theta_0} \equiv \mathrm{E}_{\mathrm{T}\theta_0} \frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0^2} \right),$$

$$\frac{\partial^2 \hat{\overline{l}}_{\theta_0}}{\partial \theta_0^2} = \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0^2} \right)_{O_p(1)} + \left\{ \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right)_{O_p(1)} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})} \right\}_{O_p(N^{-1/2})}$$

$$+ \left[\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right)_{O_p(1)} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1})} \right.$$

$$\left. + \left(\frac{1}{2} \frac{\partial^4 \overline{l}_{\theta_0}}{\partial \theta_0^2 (\partial \boldsymbol{\alpha}_0 ')^{<2>}} \right)_{O_p(1)} \{ (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>} \}_{O_p(N^{-1})} \right]_{O_p(N^{-1})} + O_p(N^{-3/2})$$

$$= \lambda_{\theta_0} + \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0^2} - \lambda_{\theta_0} \right)_{O_p(n^{-1/2})}$$

$$+ \left\{ \left(\mathrm{E}_{\mathrm{T}\theta_0} \frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right)_{O(1)} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})} \right\}_{O_p(N^{-1/2})}$$

$$+ \left\{ \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0 '} - \mathrm{E}_{\mathrm{T}\theta_0}(\cdot) \right)_{O_p(n^{-1/2})} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})} \right\}_{O_p(n^{-1/2} N^{-1/2})}$$

$$+ \left[\left(\mathrm{E}_{\mathrm{T}\theta_0} \frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right)_{O(1)} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(2)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(2)} - N^{-1} \boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1})} \right.$$

$$\left. + \left(\frac{1}{2} \mathrm{E}_{\mathrm{T}\theta_0} \frac{\partial^4 \overline{l}_{\theta_0}}{\partial \theta_0^2 (\partial \boldsymbol{\alpha}_0 ')^{<2>}} \right)_{O(1)} \{ (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>} \}_{O_p(N^{-1})} \right]_{O_p(N^{-1})}$$

$$+ O_p(n^{-1/2} N^{-1}) + O_p(N^{-3/2}),$$

$$-\hat{\lambda}_{\theta_0} = -\lambda_{\theta_0} - \left(\frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0} \mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0} \right)_{O_p(N^{-1/2})} - \left\{ \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} (\boldsymbol{\Gamma}^{(2)}_{\boldsymbol{\alpha}_0} \mathbf{l}^{(2)}_{\boldsymbol{\alpha}_0} - N^{-1} \boldsymbol{\Lambda}^{-1}_{\boldsymbol{\alpha}_0} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0}) + \left(\frac{1}{2} \mathrm{E}_{\mathrm{T}\theta_0} \frac{\partial^2 \lambda_{\theta_0}}{(\partial \boldsymbol{\alpha}_0 ')^{<2>}} \right) (\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0} \mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})^{<2>} \right\}_{O_p(N^{-1})} + O_p(N^{-3/2}),$$

where note that under m.m. $\mathrm{E}_{\mathrm{T}\theta_0} \frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0} = \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0}$, whereas under c.m.s. this does not hold since $P_{\mathrm{T}k}(\cdot) = P_k(\cdot)\ (k = 1, \ldots, n)$ are functions of $\boldsymbol{\alpha}$. Similar properties are also found for $\mathrm{E}_{\mathrm{T}\theta_0} \frac{\partial^4 \overline{l}_{\theta_0}}{\partial \theta_0^2 (\partial \boldsymbol{\alpha}_0)^{<2>}}$ and $\frac{\partial^2 \lambda_{\theta_0}}{(\partial \boldsymbol{\alpha}_0)^{<2>}}$. Then, it follows that

$$\hat{m} = \frac{\partial^2 \hat{\overline{l}}_{\theta_0}}{\partial \theta_0^2} - \hat{\lambda}_{\theta_0}$$

$$= \left(\frac{\partial^2 \overline{l}_{\theta_0}}{\partial \theta_0^2} - \lambda_{\theta_0} \right)_{O_p(n^{-1/2})}$$

$$+ \left[\left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right\}_{O(1)} (\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0} \mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1/2})} \right]_{O_p(N^{-1/2})}$$

$$+ \left\{ \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0 '} - \mathrm{E}_{\mathrm{T}\theta_0}(\cdot) \right)_{O_p(n^{-1/2})} (\boldsymbol{\Gamma}^{(1)}_{\boldsymbol{\alpha}_0} \mathbf{l}^{(1)}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1/2})} \right\}_{O_p(n^{-1/2} N^{-1/2})}$$

$$+ \left[\left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0 '} \right) - \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0 '} \right\}_{O(1)} (\boldsymbol{\Gamma}^{(2)}_{\boldsymbol{\alpha}_0} \mathbf{l}^{(2)}_{\boldsymbol{\alpha}_0} - N^{-1} \boldsymbol{\Lambda}^{-1}_{\boldsymbol{\alpha}_0} \boldsymbol{\eta}_{\boldsymbol{\alpha}_0})_{O_p(N^{-1})} \right.$$

$$+\frac{1}{2}\left\{\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^4 \overline{l}_{\theta_0}}{\partial\theta_0^2(\partial\boldsymbol{\alpha}_0')^{<2>}}\right)-\frac{\partial^2\lambda_{\theta_0}}{(\partial\boldsymbol{\alpha}_0)^{<2>}}\right\}_{O(1)}\{(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})^{<2>}\}_{O_p(N^{-1})}\Bigg]_{O_p(N^{-1})}$$

$$+O_p(n^{-1/2}N^{-1})+O_p(N^{-3/2})$$

$$\equiv m_{O_p(n^{-1/2})}+(m^{(\Delta)})_{O_p(N^{-1/2})}+(m^{(\Delta\Delta a)})_{O_p(n^{-1/2}N^{-1/2})}+(m^{(\Delta\Delta b)})_{O_p(N^{-1})}$$

$$+O_p(n^{-1/2}N^{-1})+O_p(N^{-3/2}),$$

where the terms of orders $O_p(N^{-1/2})$ and $O_p(N^{-1})$ vanish under m.m.

Using the above definitions, we have

$$\hat{\mathbf{l}}_{\theta_0}^{(2)}=[m_{O_p(n^{-1/2})}(l_{\theta_0}^{(1)})_{O_p(n^{-1/2})},\ \{(l_{\theta_0}^{(1)})^2\}_{O_p(n^{-1})}]'_{O_p(n^{-1})}$$

$$+\begin{bmatrix} m_{O_p(n^{-1/2})}(l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} & \\ +(m^{(\Delta)})_{O_p(N^{-1/2})}(l_{\theta_0}^{(1)})_{O_p(n^{-1/2})}, & 2(l_{\theta_0}^{(1)})_{O_p(n^{-1/2})}(l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})}\end{bmatrix}'_{O_p(n^{-1/2}N^{-1/2})}$$

$$+[(m^{(\Delta)})_{O_p(N^{-1/2})}(l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})},\ \{(l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}]'_{O_p(N^{-1})}$$

$$+[m_{O_p(n^{-1/2})}(l_{\theta_0}^{(\Delta\Delta a1)})_{O_p(n^{-1/2}N^{-1/2})}+(m^{(\Delta\Delta a)})_{O_p(n^{-1/2}N^{-1/2})}(l_{\theta_0}^{(1)})_{O_p(n^{-1/2})},$$

$$2(l_{\theta_0}^{(1)})_{O_p(n^{-1/2})}(l_{\theta_0}^{(\Delta\Delta a1)})_{O_p(n^{-1/2}N^{-1/2})}]'_{O_p(n^{-1}N^{-1/2})}$$

$$+[m_{O_p(n^{-1/2})}(l_{\theta_0}^{(\Delta\Delta b1)})_{O_p(N^{-1})}+(m^{(\Delta)})_{O_p(N^{-1/2})}(l_{\theta_0}^{(\Delta\Delta a1)})_{O_p(n^{-1/2}N^{-1/2})}$$

$$+(m^{(\Delta\Delta a)})_{O_p(n^{-1/2}N^{-1/2})}(l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})}+(m^{(\Delta\Delta b)})_{O_p(N^{-1})}(l_{\theta_0}^{(1)})_{O_p(n^{-1/2})},$$

$$2(l_{\theta_0}^{(1)})_{O_p(n^{-1/2})}(l_{\theta_0}^{(\Delta\Delta b1)})_{O_p(N^{-1})}+2(l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})}(l_{\theta_0}^{(\Delta\Delta a1)})_{O_p(n^{-1/2}N^{-1/2})}]'_{O_p(n^{-1/2}N^{-1})}$$

$$+[(m^{(\Delta)})_{O_p(N^{-1/2})}(l_{\theta_0}^{(\Delta\Delta b1)})_{O_p(N^{-1})}+(m^{(\Delta\Delta b)})_{O_p(N^{-1})}(l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})},$$

$$2(l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})}(l_{\theta_0}^{(\Delta\Delta b1)})_{O_p(N^{-1})}]'_{O_p(N^{-3/2})}$$

$$\equiv(\mathbf{l}_{\theta_0}^{(2)})_{O_p(n^{-1})}+(\mathbf{l}_{\theta_0}^{(\Delta a2)})_{O_p(n^{-1/2}N^{-1/2})}+(\mathbf{l}_{\theta_0}^{(\Delta b2)})_{O_p(N^{-1})}$$

$$+(\mathbf{l}_{\theta_0}^{(\Delta\Delta a2)})_{O_p(n^{-1}N^{-1/2})}+(\mathbf{l}_{\theta_0}^{(\Delta\Delta b2)})_{O_p(n^{-1/2}N^{-1})}+(\mathbf{l}_{\theta_0}^{(\Delta\Delta c2)})_{O_p(N^{-3/2})}$$

$$+O_p(n^{-1}N^{-1})+O_p(n^{-1/2}N^{-3/2})+O_p(N^{-2}).$$

(d) $\hat{\mathbf{l}}_{\theta_0}^{(3)} = \left[\hat{m}^2 \frac{\partial \hat{\bar{l}}_{\theta_0}}{\partial \theta_0},\ \hat{m} \left(\frac{\partial \hat{\bar{l}}_{\theta_0}}{\partial \theta_0} \right)^2, \left\{ \frac{\partial^3 \hat{\bar{l}}_{\theta_0}}{\partial \theta_0^3} - \mathrm{E}_{\mathrm{T}_{\theta_0}} \left(\frac{\partial^3 \hat{\bar{l}}_{\theta_0}}{\partial \theta_0^3} \right) \right\} \left(\frac{\partial \hat{\bar{l}}_{\theta_0}}{\partial \theta_0} \right)^2, \right.$

$$\left. \left(\frac{\partial \hat{\bar{l}}_{\theta_0}}{\partial \theta_0} \right)^3,\ n^{-1} \left(\hat{m},\ \frac{\partial \hat{\bar{l}}_{\theta_0}}{\partial \theta_0} \right) \right]'$$

Define $\hat{m}^{(3)} = \frac{\partial^3 \hat{\bar{l}}_{\theta_0}}{\partial \theta_0^3} - \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \hat{\bar{l}}_{\theta_0}}{\partial \theta_0^3} \right)$. Then, as in $\hat{m}$,

$$\hat{m}^{(3)} = \left\{ \frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} - \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} \right) \right\}_{O_p(n^{-1/2})}$$

$$+ \left[\left\{ \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^4 \bar{l}_{\theta_0}}{\partial \theta_0^3 \partial \boldsymbol{\alpha}_0{}'} \right) - \frac{\partial}{\partial \boldsymbol{\alpha}_0{}'} \mathrm{E}_{\mathrm{T}\theta_0} \left(\frac{\partial^3 \bar{l}_{\theta_0}}{\partial \theta_0^3} \right) \right\}_{O(1)} (\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)} \mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)})_{O_p(N^{-1/2})} \right]_{O_p(N^{-1/2})}$$

$$+ O_p(n^{-1/2} N^{-1/2}) + O_p(N^{-1})$$

$$\equiv (m^{(3)})_{O_p(n^{-1/2})} + (m^{(\Delta 3)})_{O_p(N^{-1/2})} + O_p(n^{-1/2} N^{-1/2}) + O_p(N^{-1}),$$

where the term of order $O_p(N^{-1/2})$, and the residual term $O_p(N^{-1})$ vanish under m.m. From the above definitions,

$$\hat{\mathbf{l}}_{\theta_0}^{(3)} = [(m^2)_{O_p(n^{-1})} (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})},\ m_{O_p(n^{-1/2})} \{(l_{\theta_0}^{(1)})^2\}_{O_p(n^{-1})},$$

$$(m^{(3)})_{O_p(n^{-1/2})} \{(l_{\theta_0}^{(1)})^2\}_{O_p(n^{-1})},\ \{(l_{\theta_0}^{(1)})^3\}_{O_p(n^{-3/2})}, n^{-1}(m,\ l_{\theta_0}^{(1)})_{O_p(n^{-1/2})}]'_{O_p(n^{-3/2})}$$

$$+[2\ m_{O_p(n^{-1/2})} (m^{(\Delta)})_{O_p(N^{-1/2})} (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})} + (m^2)_{O_p(n^{-1})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})},$$

$$2\ m_{O_p(n^{-1/2})} (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} + (m^{(\Delta)})_{O_p(N^{-1/2})} \{(l_{\theta_0}^{(1)})^2\}_{O_p(n^{-1})},$$

$$2 (m^{(3)})_{O_p(n^{-1/2})} (l_{\theta_0}^{(1)})_{O_p(n^{-1/2})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} + (m^{(\Delta 3)})_{O_p(N^{-1/2})} \{(l_{\theta_0}^{(1)})^2\}_{O_p(n^{-1})},$$

$$3\{(l_{\theta_0}^{(1)})^2\}_{O_p(n^{-1})} (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})},\ n^{-1}(m^{(\Delta)}, l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})}]'_{O_p(n^{-1}N^{-1/2})}$$

$$+[2\, m_{O_p(n^{-1/2})}(m^{(\Delta)})_{O_p(N^{-1/2})}(l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} + \{(m^{(\Delta)})^2\}_{O_p(N^{-1})}(l_{\theta_0}^{(1)})_{O_p(n^{-1/2})},$$

$$2(m^{(\Delta)})_{O_p(N^{-1/2})}(l_{\theta_0}^{(1)})_{O_p(n^{-1/2})}(l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} + m_{O_p(n^{-1/2})}\{(l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})},$$

$$2(m^{(\Delta 3)})_{O_p(N^{-1/2})}(l_{\theta_0}^{(1)})_{O_p(n^{-1/2})}(l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} + (m^{(3)})_{O_p(n^{-1/2})}\{(l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})},$$

$$3\{(l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}(l_{\theta_0}^{(1)})_{O_p(n^{-1/2})},\ (0,\ \ 0)]'_{O_p(n^{-1/2}N^{-1})}$$

$$+[\{(m^{(\Delta)})^2\}_{O_p(N^{-1})}(l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})}, (m^{(\Delta)})_{O_p(N^{-1/2})}\{(l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})},$$

$$(m^{(\Delta 3)})_{O_p(N^{-1/2})}\{(l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}, \{(l_{\theta_0}^{(\Delta 1)})^3\}_{O_p(N^{-3/2})},\ (0,\ 0)]'_{O_p(N^{-3/2})}$$

$$+O_p(n^{-1}N^{-1}) + O_p(n^{-1/2}N^{-3/2}) + O_p(N^{-2})$$

$$\equiv (\mathbf{l}_{\theta_0}^{(3)})_{O_p(n^{-3/2})} + (\mathbf{l}_{\theta_0}^{(\Delta a3)})_{O_p(n^{-1}N^{-1/2})} + (\mathbf{l}_{\theta_0}^{(\Delta b3)})_{O_p(n^{-1/2}N^{-1})} + (\mathbf{l}_{\theta_0}^{(\Delta c3)})_{O_p(N^{-3/2})}$$

$$+O_p(n^{-1}N^{-1}) + O_p(n^{-1/2}N^{-3/2}) + O_p(N^{-2}).$$

(e) $-(n^{-1}\hat{\lambda}_{\theta_0}^{-1}\hat{\eta}_{\theta_0})$

$$= -(n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})_{O(n^{-1})}$$

$$-\left\{ n^{-1}\left(-\lambda_{\theta_0}^{-2}\eta_{\theta_0}\frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0{}'} + \lambda_{\theta_0}^{-1}\frac{\partial \eta_{\theta_0}}{\partial \boldsymbol{\alpha}_0{}'} \right)(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}_0}^{(1)}) \right\}_{O_p(n^{-1}N^{-1/2})} + O_p(n^{-1}N^{-1})$$

$$\equiv -(n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})_{O(n^{-1})} - \{n^{-1}(\lambda_{\theta_0}^{-1}\eta_{\theta_0})^{(\Delta)}\}_{O_p(n^{-1}N^{-1/2})} + O_p(n^{-1}N^{-1}).$$

A.3 Asymptotic cumulants of $\hat{\theta}$: Proofs of Theorems 1, 2 and 3

Define $\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\cdot)$ similarly to $\mathrm{E}_{\mathrm{T}\theta_0}(\cdot)$ under p.m.m., where the true marginal multinomial distribution for 2^n response patterns is employed for the expectation associated with item calibration. Denote the two-fold expectation $\mathrm{E}_{\mathrm{T}\theta_0}\{\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\cdot)\}$ by $\mathrm{E}_{\mathrm{T}}(\cdot)$ for simplicity. The notation like $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]}$ is for ease of finding correspondence.

(a) Proof of Theorem 1 under Condition A:
$N = O(n)$ $(\bar{c} = n / N = O(1))$

Recall $w = n^{1/2}(\hat{\theta} - \theta_0)$. Then, from (A.1) the following results are obtained and give Theorem 1.

$$\kappa_1(w) = n^{1/2}\{\mathrm{E}_{\mathrm{T}}(q^{(2)}_{O_p(n^{-1})}) - (n^{-1}\lambda^{-1}_{\theta_0}\eta_{\theta_0})_{O(n^{-1})}\}_{O(n^{-1})} + O(n^{-3/2})$$

$$= n^{1/2}[\{\mathrm{E}_{\mathrm{T}\theta_0}(q^{(20)}_{O_p(n^{-1})}) - (n^{-1}\lambda^{-1}_{\theta_0}\eta_{\theta_0})\} + \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(q^{(22)}_{O_p(N^{-1})})]_{O(n^{-1})} + O(n^{-3/2})$$

$$\equiv n^{-1/2}(\beta^{(0)}_1 + \bar{\beta}^{(\Delta)}_1) + O(n^{-3/2})$$

$$\equiv n^{-1/2}\bar{\beta}_1 + O(n^{-3/2})$$

$$\equiv n^{-1/2}\beta^{(0)}_1 + N^{-1/2}\bar{c}^{1/2}\beta^{(\Delta)}_1 + O(n^{-3/2}),$$

where $\bar{c}\beta^{(\Delta)}_1 = \bar{\beta}^{(\Delta)}_1$.

$$\kappa_2(w) = n\mathrm{E}_{\mathrm{T}}[\{(q^{(1)}_{O_p(n^{-1/2})})^2\}_{O_p(n^{-1})}] + n\mathrm{E}_{\mathrm{T}}[\ \{(q^{(2)}_{O_p(n^{-1})})\}_{O_p(n^{-2})}$$

$$+ 2(q^{(1)}_{O_p(n^{-1/2})}q^{(2)}_{O_p(n^{-1})})_{O_p(n^{-3/2})} + 2(q^{(1)}_{O_p(n^{-1/2})}q^{(3)}_{O_p(n^{-3/2})})_{O_p(n^{-2})}\ ]$$

$$- n^{-1}(\beta^{(0)}_1 + \bar{\beta}^{(\Delta)}_1 + \lambda^{-1}_{\theta_0}\eta_{\theta_0})^2 + O(n^{-2})$$

$$= \{n\mathrm{E}_{\mathrm{T}\theta_0}[\{(q^{(10)}_{O_p(n^{-1/2})})^2\}_{O_p(n^{-1})}]\}_{O(1)} + \{n\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}[\{(q^{(11)}_{O_p(N^{-1/2})})^2\}_{O_p(N^{-1})}]\}_{O(nN^{-1})}$$

$$+[\ n\mathrm{E}_{\mathrm{T}\theta_0}[\{(q^{(20)}_{O_p(n^{-1})})^2\}_{O_p(n^{-2})} + 2q^{(10)}_{O_p(n^{-1/2})}q^{(20)}_{O_p(n^{-1})} + 2q^{(10)}_{O_p(n^{-1/2})}q^{(30)}_{O_p(n^{-3/2})}]$$

$$- n^{-1}(\beta^{(0)}_1 + \lambda^{-1}_{\theta_0}\eta_{\theta_0})^2\quad]_{O(n^{-1})}$$

$$+ \underset{(\mathrm{A})}{[}\ \{n\{\mathrm{E}_{\mathrm{T}}\{(q^{(21)}_{O_p(n^{-1/2}N^{-1/2})})^2 + 2q^{(20)}_{O_p(n^{-1})}q^{(22)}_{O_p(N^{-1})}\}\}_{O(N^{-1})}$$

$$+ \{n\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(22)}_{O_p(N^{-1})})^2\}\}_{O(nN^{-2})} + \{2n\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(q^{(11)}_{O_p(N^{-1/2})}q^{(22)}_{O_p(N^{-1})})\}_{O(nN^{-2})}$$

$$+\{2n\mathrm{E}_{\mathrm{T}}[\ q^{(10)}_{O_p(n^{-1/2})}q^{(32)}_{O_p(n^{-1/2}N^{-1})} + q^{(11)}_{O_p(N^{-1/2})}\{q^{(31)}_{O_p(n^{-1}N^{-1/2})}$$

$$- (n^{-1}(\lambda^{-1}_{\theta_0}\eta_{\theta_0})^{(\Delta)})_{O_p(n^{-1}N^{-1/2})}\}\ \ ]\}_{O(N^{-1})}$$

$$+\{2n\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(q^{(11)}_{O_p(N^{-1/2})}q^{(33)}_{O_p(N^{-3/2})})\}_{O(nN^{-2})}$$

$$- n^{-1}\{2\bar{\beta}^{(\Delta)}_1(\beta^{(0)}_1 + \lambda^{-1}_{\theta_0}\eta_{\theta_0}) + (\bar{\beta}^{(\Delta)}_1)^2\}\quad]_{(\mathrm{A})O(n^{-1})} + O(n^{-2})$$

$$\equiv \beta_2^{(0)} + \overline{\beta}_2^{(\Delta)} + n^{-1}\beta_{\mathrm{H}2}^{(0)} + n^{-1}\overline{\beta}_{\mathrm{H}2}^{(\Delta)} + O(n^{-2})$$
$$\equiv \overline{\beta}_2 + n^{-1}\overline{\beta}_{\mathrm{H}2} + O(n^{-2})$$
$$\equiv \beta_2^{(0)} + \overline{c}\beta_2^{(\Delta)} + n^{-1}\beta_{\mathrm{H}2}^{(0)} + N^{-1}\beta_{\mathrm{H}2}^{(\Delta a)} + N^{-1}\overline{c}\beta_{\mathrm{H}2}^{(\Delta b)} + O(n^{-2}),$$

where $\overline{\beta}_2 = \beta_2^{(0)} + \overline{\beta}_2^{(\Delta)}$, $\overline{\beta}_2^{(\Delta)} = \overline{c}\beta_2^{(\Delta)}, \overline{\beta}_{\mathrm{H}2}^{(\Delta)} = \overline{c}\beta_{\mathrm{H}2}^{(\Delta a)} + \overline{c}^2\beta_{\mathrm{H}2}^{(\Delta b)}$; $N^{-1}\beta_{\mathrm{H}2}^{(\Delta a)}$ and $N^{-1}\overline{c}\beta_{\mathrm{H}2}^{(\Delta b)}$ denote the sums of orders $O(N^{-1})$ and $O(nN^{-2})$ inside $\underset{(\mathrm{A})}{[}\cdot\underset{(\mathrm{A})}{]}$. Since the constant term with no expectations in $\underset{(\mathrm{A})}{[}\cdot\underset{(\mathrm{A})}{]}$ is $-n^{-1}\{2\overline{\beta}_1^{(\Delta)}(\beta_1^{(0)} + \lambda_{\theta_0}^{-1}\eta_{\theta_0}) + (\overline{\beta}_1^{(\Delta)})^2\}$ $= -N^{-1}2\beta_1^{(\Delta)}(\beta_1^{(0)} + \lambda_{\theta_0}^{-1}\eta_{\theta_0}) - N^{-1}\overline{c}(\beta_1^{(\Delta)})^2$, the two terms on the right-hand side of the last equation are included in $N^{-1}\beta_{\mathrm{H}2}^{(\Delta a)}$ and $N^{-1}\overline{c}\beta_{\mathrm{H}2}^{(\Delta b)}$, respectively.

$$\kappa_3(w) = n^{3/2}[\ \{\mathrm{E_T}\{(q^{(1)}_{O_p(n^{-1/2})})^3 + 3(q^{(1)}_{O_p(n^{-1/2})})^2 q^{(2)}_{O_p(n^{-1})}\}\}_{O(n^{-2})}$$
$$- 3n^{-2}(\beta_1^{(0)} + \overline{\beta}_1^{(\Delta)} + \lambda_{\theta_0}^{-1}\eta_{\theta_0})(\beta_2^{(0)} + \overline{\beta}_2^{(\Delta)})\ ]_{O(n^{-2})} + O(n^{-3/2})$$
$$= n^{3/2}[\ \{\mathrm{E}_{\mathrm{T}\theta_0}\{(q^{(10)}_{O_p(n^{-1/2})})^3 + 3(q^{(10)}_{O_p(n^{-1/2})})^2 q^{(20)}_{O_p(n^{-1})}\}\}_{O(n^{-2})}$$
$$- 3n^{-2}(\beta_1^{(0)} + \lambda_{\theta_0}^{-1}\eta_{\theta_0})\beta_2^{(0)}\ ]_{O(n^{-2})}$$
$$+ \underset{(\mathrm{A})}{[}\ [n^{3/2}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(11)}_{O_p(N^{-1/2})})^3\}]_{O(n^{3/2}N^{-2})}$$
$$+[3n^{3/2}\mathrm{E_T}\{(q^{(10)}_{O_p(n^{-1/2})})^2 q^{(22)}_{O_p(N^{-1})} + 2q^{(10)}_{O_p(n^{-1/2})}q^{(11)}_{O_p(N^{-1/2})}q^{(21)}_{O_p(n^{-1/2}N^{-1/2})}$$
$$+ (q^{(11)}_{O_p(N^{-1/2})})^2 q^{(20)}_{O_p(n^{-1})}\}]_{O(n^{1/2}N^{-1})}$$
$$+[3n^{3/2}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(11)}_{O_p(N^{-1/2})})^2 q^{(22)}_{O_p(N^{-1})}\}]_{O(n^{3/2}N^{-2})}$$
$$- 3n^{-1/2}\{(\beta_1^{(0)} + \overline{\beta}_1^{(\Delta)} + \lambda_{\theta_0}^{-1}\eta_{\theta_0})\overline{\beta}_2^{(\Delta)} + \overline{\beta}_1^{(\Delta)}\beta_2^{(0)}\}\ \underset{(\mathrm{A})}{]}_{O(n^{-1/2})} + O(n^{-3/2})$$
$$\equiv n^{-1/2}(\beta_3^{(0)} + \overline{\beta}_3^{(\Delta)}) + O(n^{-3/2})$$
$$\equiv n^{-1/2}\overline{\beta}_3 + O(n^{-3/2})$$
$$\equiv n^{-1/2}\beta_3^{(0)} + N^{-1/2}(\overline{c}^{1/2}\beta_3^{(\Delta a)} + \overline{c}^{3/2}\beta_3^{(\Delta b)}) + O(n^{-3/2}),$$

where $\overline{\beta}_3^{(\Delta)} = \overline{c}\beta_3^{(\Delta a)} + \overline{c}^2\beta_3^{(\Delta b)}$. The terms $N^{-1/2}\overline{c}^{1/2}\beta_3^{(\Delta a)}$ and $N^{-1/2}\overline{c}^{3/2}\beta_3^{(\Delta b)}$ shown above denote the sums of the terms of orders $O(n^{1/2}N^{-1})$ (given by $\mathrm{E}_{\mathrm{T}}(\cdot)$) and $O(n^{3/2}N^{-2})$ (given by $\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(\cdot)$), respectively. Since the constant terms with no expectations in $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]}$ are

$$-3n^{-1/2}\overline{c}\{(\beta_1^{(0)} + \lambda_{\theta_0}^{-1}\eta_{\theta_0})\beta_2^{(\Delta)} + \beta_1^{(\Delta)}\beta_2^{(0)}\} - 3n^{-1/2}\overline{c}^2\beta_1^{(\Delta)}\beta_2^{(\Delta)}$$
$$= -3N^{-1/2}\overline{c}^{1/2}\{(\beta_1^{(0)} + \lambda_{\theta_0}^{-1}\eta_{\theta_0})\beta_2^{(\Delta)} + \beta_1^{(\Delta)}\beta_2^{(0)}\} - 3N^{-1/2}\overline{c}^{3/2}\beta_1^{(\Delta)}\beta_2^{(\Delta)},$$

the two terms on the right-hand side of the equation are included in $N^{-1/2}\overline{c}^{1/2}\beta_3^{(\Delta a)}$ and $N^{-1/2}\overline{c}^{3/2}\beta_3^{(\Delta b)}$, respectively.

Define $\overline{\beta}_1^* = \overline{\beta}_1 + \lambda_{\theta_0}^{-1}\eta_{\theta_0}$. Then,

$$\kappa_4(w) = n^2[\ \{\mathrm{E}_{\mathrm{T}}\{(q^{(1)}_{O_p(n^{-1/2})})^4 - 3n^{-2}\overline{\beta}_2^2 + 4(q^{(1)}_{O_p(n^{-1/2})})^3 q^{(2)}_{O_p(n^{-1})}$$
$$+ 6(q^{(1)}_{O_p(n^{-1/2})})^2(q^{(2)}_{O_p(n^{-1})})^2 + 4(q^{(1)}_{O_p(n^{-1/2})})^3 q^{(3)}_{O_p(n^{-3/2})}\}\}_{O(n^{-3})}$$
$$-4n^{-3}\overline{\beta}_1^*(\overline{\beta}_3 + 3\overline{\beta}_1^*\overline{\beta}_2) - 6n^{-3}\overline{\beta}_2\overline{\beta}_{\mathrm{H}2} + 6n^{-3}\overline{\beta}_2(\overline{\beta}_1^*)^2\quad]_{O(n^{-3})} + O(n^{-2})$$
$$= \underset{(\mathrm{A})}{[}\ n^2\ \{\mathrm{E}_{\mathrm{T}\theta_0}\{(q^{(10)}_{O_p(n^{-1/2})})^4 - 3n^{-2}(\beta_2^{(0)})^2 + 4(q^{(10)}_{O_p(n^{-1/2})})^3 q^{(20)}_{O_p(n^{-1})}$$
$$+ 6(q^{(10)}_{O_p(n^{-1/2})})^2(q^{(20)}_{O_p(n^{-1})})^2 + 4(q^{(10)}_{O_p(n^{-1/2})})^3 q^{(30)}_{O_p(n^{-3/2})}\}\}_{O(n^{-3})}$$
$$- 4n^{-1}\beta_1^{(0)}\beta_3^{(0)} - 6n^{-1}\beta_2^{(0)}\beta_{\mathrm{H}2}^{(0)} - 6n^{-1}\beta_2^{(0)}(\beta_1^{(0)})^2\quad \underset{(\mathrm{A})}{]}_{O(n^{-1})}$$

(the following 9 terms are numbered as Terms (1) to (9))

$$+ \underset{(\mathrm{B})}{[}\ \{6n^2\,\mathrm{E}_{\mathrm{T}}\{(q^{(10)}_{O_p(n^{-1/2})}q^{(11)}_{O_p(N^{-1/2})})^2 - n^{-2}\beta_2^{(0)}\overline{\beta}_2^{(\Delta)}\}\}_0$$
$$+ \{n^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(11)}_{O_p(N^{-1/2})})^4 - 3n^{-2}(\overline{\beta}_2^{(\Delta)})^2\}\}_{O(n^2N^{-3})}$$
$$+ \{4n^2\mathrm{E}_{\mathrm{T}}\{(q^{(10)}_{O_p(n^{-1/2})})^3 q^{(22)}_{O_p(N^{-1})} + 3(q^{(10)}_{O_p(n^{-1/2})})^2 q^{(11)}_{O_p(N^{-1/2})}$$
$$\times(q^{(21)}_{O_p(n^{-1/2}N^{-1/2})} + q^{(22)}_{O_p(N^{-1})})\}\}_{O(N^{-1})+O(nN^{-2})}$$
$$+\{4n^2\mathrm{E}_{\mathrm{T}}\{3q^{(10)}_{O_p(n^{-1/2})}(q^{(11)}_{O_p(N^{-1/2})})^2(q^{(20)}_{O_p(n^{-1})} + q^{(21)}_{O_p(n^{-1/2}N^{-1/2})})\}\}_{O(N^{-1})+O(nN^{-2})}$$
$$+\{4n^2\mathrm{E}_{\mathrm{T}}\{(q^{(11)}_{O_p(N^{-1/2})})^3(q^{(20)}_{O_p(n^{-1})} + q^{(22)}_{O_p(N^{-1})})\}\}_{O(nN^{-2})+O(n^2N^{-3})}$$

$$+\{6n^2\mathrm{E_T}\{(q^{(10)}_{O_p(n^{-1/2})})^2((q^{(21)}_{O_p(n^{-1/2}N^{-1/2})})^2+(q^{(22)}_{O_p(N^{-1})})^2$$

$$+2q^{(20)}_{O_p(n^{-1})}q^{(22)}_{O_p(N^{-1})})\}\}_{O(N^{-1})+O(nN^{-2})}$$

$$+\{6n^2\mathrm{E_T}\{2q^{(10)}_{O_p(n^{-1/2})}q^{(11)}_{O_p(N^{-1/2})}2q^{(21)}_{O_p(n^{-1/2}N^{-1/2})}(q^{(20)}_{O_p(n^{-1})}+q^{(22)}_{O_p(N^{-1})})\}\}_{O(N^{-1})+O(nN^{-2})}$$

$$+\{6n^2\mathrm{E_T}\{(q^{(11)}_{O_p(N^{-1/2})})^2((q^{(20)}_{O_p(n^{-1})})^2+(q^{(21)}_{O_p(n^{-1/2}N^{-1/2})})^2$$

$$+2q^{(20)}_{O_p(n^{-1})}q^{(22)}_{O_p(N^{-1})})\}\}_{O(N^{-1})+O(nN^{-2})}$$

$$+\{6n^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(11)}_{O_p(N^{-1/2})})^2(q^{(22)}_{O_p(N^{-1})})^2\}\}_{O(n^2N^{-3})}$$

(the following 5 terms are numbered as Terms (10) to (14))

$$+\{4n^2\mathrm{E_T}\{(q^{(10)}_{O_p(n^{-1/2})})^3q^{(32)}_{O_p(n^{-1/2}N^{-1})}\}\}_{O(N^{-1})}$$

$$+\{4n^2\mathrm{E_T}\{3(q^{(10)}_{O_p(n^{-1/2})})^2q^{(11)}_{O_p(N^{-1/2})}(q^{(31)}_{O_p(n^{-1}N^{-1/2})}+q^{(33)}_{O_p(N^{-3/2})}$$

$$-(n^{-1}(\lambda^{-1}_{\theta_0}\eta_{\theta_0})^{(\Delta)})_{O_p(n^{-1}N^{-1/2})})\}\}_{O(N^{-1})+O(nN^{-2})}$$

$$+\{4n^2\mathrm{E_T}\{3q^{(10)}_{O_p(n^{-1/2})}(q^{(11)}_{O_p(N^{-1/2})})^2(q^{(30)}_{O_p(n^{-3/2})}+q^{(32)}_{O_p(n^{-1/2}N^{-1})})\}\}_{O(N^{-1})+O(nN^{-2})}$$

$$+\{4n^2\mathrm{E_T}\{(q^{(11)}_{O_p(N^{-1/2})})^3(q^{(31)}_{O_p(n^{-1}N^{-1/2})}-(n^{-1}(\lambda^{-1}_{\theta_0}\eta_{\theta_0})^{(\Delta)})_{O_p(n^{-1}N^{-1/2})})\}\}_{O(nN^{-2})}$$

$$+\{4n^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(11)}_{O_p(N^{-1/2})})^3q^{(33)}_{O_p(N^{-3/2})}\}\}_{O(n^2N^{-3})}$$

$$-4n^{-1}\{(\beta^{(0)}_1+\lambda^{-1}_{\theta_0}\eta_{\theta_0})\bar\beta^{(\Delta)}_3+\bar\beta^{(\Delta)}_1(\beta^{(0)}_3+\bar\beta^{(\Delta)}_3)\}$$

$$-6n^{-1}\{\beta^{(0)}_2\bar\beta^{(\Delta)}_{\mathrm{H}2}+\bar\beta^{(\Delta)}_2(\beta^{(0)}_{\mathrm{H}2}+\bar\beta^{(\Delta)}_{\mathrm{H}2})\}$$

$$-6n^{-1}\{\bar\beta^{(\Delta)}_2(\beta^{(0)}_1+\lambda^{-1}_{\theta_0}\eta_{\theta_0}+\bar\beta^{(\Delta)}_1)^2+\beta^{(0)}_2\{2(\beta^{(0)}_1+\lambda^{-1}_{\theta_0}\eta_{\theta_0})\bar\beta^{(\Delta)}_1$$

$$+(\bar\beta^{(\Delta)}_1)^2\}\}\quad]_{(\mathrm{B})O(n^{-1})}+O(n^{-2})$$

$$\equiv n^{-1}(\beta^{(0)}_4+\bar\beta^{(\Delta)}_4)+O(n^{-2}),$$

where Term (1) is 0 and Term (2) is

$$n^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}[\{(\gamma^{(1)}_{\theta_0}l^{(\Delta1)}_{\theta_0})^4\}_{O_p(N^{-2})}]-3(\bar\beta^{(\Delta)}_2)^2$$

$$=\bar c^2[N^2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}[\{(\gamma^{(1)}_{\theta_0}l^{(\Delta1)}_{\theta_0})^4\}_{O_p(N^{-2})}]-3(\beta^{(\Delta)}_2)^2]$$

$$=N^{-1}\bar c^2\{N^3\kappa_4(\gamma^{(1)}_{\theta_0}l^{(\Delta1)}_{\theta_0})\}_{O(1)}.$$

Define $\beta_1^{(0)*} = \beta_1^{(0)} + \lambda_{\theta_0}^{-1}\eta_{\theta_0}$. Then, the constant term with no expectations in $\underset{(\mathrm{B})}{[}\cdot\underset{(\mathrm{B})}{]}$ is

$$-4n^{-1}\{(\beta_1^{(0)*}+\bar{c}\beta_1^{(\Delta)})(\bar{c}\beta_3^{(\Delta a)}+\bar{c}^2\beta_3^{(\Delta b)})+\bar{c}\beta_1^{(\Delta)}\beta_3^{(0)}\}$$
$$-6n^{-1}\{(\beta_2^{(0)}+\bar{c}\beta_2^{(\Delta)})(\bar{c}\beta_{\mathrm{H}2}^{(\Delta a)}+\bar{c}^2\beta_{\mathrm{H}2}^{(\Delta b)})+\bar{c}\beta_2^{(\Delta)}\beta_{\mathrm{H}2}^{(0)}\}$$
$$-6n^{-1}[\bar{c}\beta_2^{(\Delta)}(\beta_1^{(0)*}+\bar{c}\beta_1^{(\Delta)})^2+\beta_2^{(0)}\{2\bar{c}\beta_1^{(0)*}\beta_1^{(\Delta)}+\bar{c}^2(\beta_1^{(\Delta)})^2\}]$$
$$\begin{aligned}=-N^{-1}[\quad&\{4\beta_1^{(0)*}\beta_3^{(\Delta a)}+4\beta_1^{(\Delta)}\beta_3^{(0)}+6\beta_2^{(0)}\beta_{\mathrm{H}2}^{(\Delta a)}+6\beta_2^{(\Delta)}\beta_{\mathrm{H}2}^{(0)}\\&\quad+6\beta_2^{(\Delta)}(\beta_1^{(0)*})^2+12\beta_2^{(0)}\beta_1^{(0)*}\beta_1^{(\Delta)}\}\\&+\bar{c}\{4\beta_1^{(0)*}\beta_3^{(\Delta b)}+4\beta_1^{(\Delta)}\beta_3^{(\Delta a)}+6\beta_2^{(\Delta)}\beta_{\mathrm{H}2}^{(\Delta a)}+6\beta_2^{(0)}\beta_{\mathrm{H}2}^{(\Delta b)}\\&\quad+12\beta_2^{(\Delta)}\beta_1^{(0)*}\beta_1^{(\Delta)}+6\beta_2^{(0)}(\beta_1^{(\Delta)})^2\}\\&+\bar{c}^2\{4\beta_1^{(\Delta)}\beta_3^{(\Delta b)}+6\beta_2^{(\Delta)}\beta_{\mathrm{H}2}^{(\Delta b)}+6\beta_2^{(\Delta)}(\beta_1^{(\Delta)})^2\}\quad]\end{aligned}$$

(note that $n^{-1} = N^{-1}\bar{c}^{-1}$ and $N^{-1} = n^{-1}\bar{c}$).

The above results are summarized as

$$\begin{aligned}\kappa_4(w) &= n^{-1}(\beta_4^{(0)}+\bar{\beta}_4^{(\Delta)})+O(n^{-2})\\&\equiv n^{-1}\beta_4^{(0)}+N^{-1}(\beta_4^{(\Delta a)}+\bar{c}\beta_4^{(\Delta b)}+\bar{c}^2\beta_4^{(\Delta c)})+O(n^{-2}).\end{aligned}$$

The terms, except Term (2), other than the constant terms are included in $N^{-1}\beta_4^{(\Delta a)}$ for the terms of order $O(N^{-1})$, $N^{-1}\bar{c}\beta_4^{(\Delta b)}$ for the terms of order $O(nN^{-2})$, and $N^{-1}\bar{c}^2\beta_4^{(\Delta c)}$ for the terms of order $O(n^2N^{-3})$. The cumulant term or Term (2) is included in $N^{-1}\bar{c}^2\beta_4^{(\Delta c)}$. The constants have terms included in $N^{-1}\beta_4^{(\Delta a)}$, $N^{-1}\bar{c}\beta_4^{(\Delta b)}$ and $N^{-1}\bar{c}^2\beta_4^{(\Delta c)}$.

(b) Proof of Theorem 2 under Condition B: $N = O(n^{3/2})$ $(\bar{c}^* = n^{3/2}/N = O(1))$

From (A.2), the following results giving Theorem 2 are obtained.

$$\kappa_1(w) = n^{1/2}\{\mathrm{E_T}(q^{(2)}_{O_p(n^{-1})}) - (n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})_{O(n^{-1})}\} + O(n^{-1})$$
$$= n^{1/2}[\mathrm{E}_{\mathrm{T}\theta_0}\{(\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\mathbf{l}_{\theta_0}^{(2)})_{O_p(n^{-1})}\}]_{O(n^{-1})} - n^{-1/2}\lambda_{\theta_0}^{-1}\eta_{\theta_0} + O(n^{-1})$$
$$= n^{-1/2}\beta_1^{(0)} + O(n^{-1}),$$

where $\bar{\beta}_1 = \beta_1^{(0)} = n\mathrm{E}_{\mathrm{T}\theta_0}\{(\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\mathbf{l}_{\theta_0}^{(2)})_{O_p(n^{-1})}\} - \lambda_{\theta_0}^{-1}\eta_{\theta_0}$ $(\bar{\beta}_1^{(\Delta)} = 0)$ (the order of the residual is not $O(n^{-3/2})$ but $O(n^{-1})$, which is due to the term of order $O_p(N^{-1}) = O_p(n^{-3/2})$ in (A.2)).

$$\kappa_2(w) = n\mathrm{E}_{\mathrm{T}\theta_0}\{(q_{O_p(n^{-1/2})}^{(10)})^2\} + n\mathrm{E}_{\mathrm{T}}\{(q_{O_p(N^{-1/2})}^{(1a)})^2\} + n\mathrm{E}_{\mathrm{T}\theta_0}\{(q_{O_p(n^{-1})}^{(20)})^2$$
$$+ 2q_{O_p(n^{-1/2})}^{(10)}(q_{O_p(n^{-1})}^{(20)} + q_{O_p(n^{-3/2})}^{(30)})\} - n^{-1}(\beta_1^{(0)*})^2 + O(n^{-3/2})$$
$$= \beta_2^{(0)} + n\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}[\{(\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}] + n^{-1}\beta_{\mathrm{H}2}^{(0)} + O(n^{-3/2})$$
$$\equiv \beta_2^{(0)} + n^{-1/2}\bar{\beta}_{\mathrm{h}2}^{(\Delta)} + n^{-1}\beta_{\mathrm{H}2}^{(0)} + O(n^{-3/2})$$
$$\equiv \beta_2^{(0)} + n^{-1/2}\bar{c}^*\beta_{\mathrm{h}2}^{(\Delta)} + n^{-1}\beta_{\mathrm{H}2}^{(0)} + O(n^{-3/2}),$$

where $\bar{\beta}_{\mathrm{h}2}^{(\Delta)} = n^{3/2}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}[\{(\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}] = \bar{c}^*\beta_{\mathrm{h}2}^{(\Delta)} = O(1) > 0$ and $\bar{\beta}_2 = \beta_2^{(0)}$ (the order $O(n^{-3/2})$ rather than $O(n^{-2})$ of the residual is due to $n\{(q_{O_p(n^{-1/2}N^{-1/2})}^{(2a)})_{O_p(n^{-5/4})}\}^2$).

$$\kappa_3(w) = [\ n^{3/2}[\ \{\mathrm{E}_{\mathrm{T}\theta_0}\{(q_{O_p(n^{-1/2})}^{(10)})^3\}\}_{O(n^{-2})}$$
$$+ 3\mathrm{E}_{\mathrm{T}\theta_0}\{(q_{O_p(n^{-1/2})}^{(10)})^2 q_{O_p(n^{-1})}^{(20)}\}\}_{O(n^{-2})} - 3n^{-2}\beta_1^{(0)*}\beta_2^{(0)}]\ ]_{O(n^{-1/2})} + O(n^{-1})$$
$$= n^{-1/2}\beta_3^{(0)} + O(n^{-1}),$$

where $\bar{\beta}_3 = \beta_3^{(0)}$ $(\bar{\beta}_3^{(\Delta)} = 0)$ (the order $O(n^{-1})$ rather than $O(n^{-3/2})$ of the residual is due to $n^{3/2}(-3n^{-5/2}\beta_1^{(0)}\bar{\beta}_{\mathrm{h}2}^{(\Delta)})$ and $3n^{3/2}\mathrm{E}_{\mathrm{T}}\{(q_{O_p(N^{-1/2})}^{(1a)})^2 q_{O_p(n^{-1})}^{(20)}\}$).

Denote temporarily the expressions $q_{O_p(\cdot)}^{(\cdot)}$ by $q^{(\cdot)}$. Then,

$$\kappa_4(w) = n^2[\ \mathrm{E}_{\mathrm{T}}\{(q^{(1)})^4 + 6(q^{(1)})^2(q^{(1a)})^2 + (q^{(1a)})^4 + 4(q^{(1)})^3 q^{(2)}$$
$$+ 6(q^{(1)})^2(q^{(2)})^2 + 4(q^{(1)})^3 q^{(3)}\}$$
$$-3n^{-2}(\beta_2^{(0)})^2 - 6n^{-5/2}\beta_2^{(0)}\bar{\beta}_{\mathrm{h}2}^{(\Delta)} - 6n^{-3}\beta_2^{(0)}\beta_{\mathrm{H}2}^{(0)} - 3n^{-3}(\bar{\beta}_{\mathrm{h}2}^{(\Delta)})^2$$
$$-4n^{-3}\beta_1^{(0)*}\beta_3^{(0)} - 6n^{-3}\beta_2^{(0)}(\beta_1^{(0)*})^2\ \ ]_{O(n^{-3})} + O(n^{-3/2})$$

$$= \underset{(A)}{[}\ n^2 \underset{(B)}{[}\ [E_{T\theta_0}\{(q^{(10)})^4\}]_{O(n^{-2})} + 4[E_{T\theta_0}\{(q^{(10)})^3 q^{(20)}\}]_{O(n^{-3})}$$

$$+6[E_{T\theta_0}\{(q^{(10)})^2(q^{(20)})^2\}]_{O(n^{-3})} \underline{+6n^{-5/2}\beta_2^{(0)}\bar{\beta}_{h2}^{(\Delta)}} \underline{+3n^{-3}(\bar{\beta}_{h2}^{(\Delta)})^2}$$

$$+4[E_{T\theta_0}\{(q^{(10)})^3 q^{(30)}\}]_{O(n^{-3})} - 3n^{-2}(\beta_2^{(0)})^2 \underline{-6n^{-5/2}\beta_2^{(0)}\bar{\beta}_{h2}^{(\Delta)}}$$

$$\underline{-3n^{-3}(\bar{\beta}_{h2}^{(\Delta)})^2} - 6n^{-3}\beta_2^{(0)}\beta_{H2}^{(0)} - 4n^{-3}\beta_1^{(0)*}\beta_3^{(0)}$$

$$-6n^{-3}\beta_2^{(0)}(\beta_1^{(0)*})^2 \quad \underset{(B)}{]} \quad \underset{(A)}{]}_{O(n^{-1})} + O(n^{-3/2})$$

$$= n^{-1}\beta_4^{(0)} + O(n^{-3/2}),$$

where the sum of the underscored terms is zero, and $\bar{\beta}_4 = \beta_4^{(0)}\ (\bar{\beta}_4^{(\Delta)} = 0)$.

(c) Proof of Theorem 3 under Condition C: $N = O(n^2)$
$(\bar{c}^{**} = n^2 / N = O(1))$

From (A.3), the following results giving Theorem 3 are obtained.

$$\kappa_1(w) = n^{1/2}\{E_T(q^{(2)}_{O_p(n^{-1})}) - (n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})_{O(n^{-1})}\} + O(n^{-3/2})$$

$$= n^{1/2}[E_{T\theta_0}\{(\boldsymbol{\gamma}_{\theta_0}^{(2)}{'}\mathbf{l}_{\theta_0}^{(2)})_{O_p(n^{-1})}\}]_{O(n^{-1})} - n^{-1/2}\lambda_{\theta_0}^{-1}\eta_{\theta_0} + O(n^{-3/2})$$

$$= n^{-1/2}\beta_1^{(0)} + O(n^{-3/2}),$$

where $\bar{\beta}_1 = \beta_1^{(0)} = nE_{T\theta_0}\{(\boldsymbol{\gamma}_{\theta_0}^{(2)}{'}\mathbf{l}_{\theta_0}^{(2)})_{O_p(n^{-1})}\} - \lambda_{\theta_0}^{-1}\eta_{\theta_0}\quad (\bar{\beta}_1^{(\Delta)} = 0)$.

$$\kappa_2(w) = nE_T\{(q^{(1)}_{O_p(n^{-1/2})})^2\} + nE_T\{(q^{(2)}_{O_p(n^{-1})})^2$$

$$+ \underline{2q^{(1)}_{O_p(n^{-1/2})}(q^{(2)}_{O_p(n^{-1})} + q^{(3)}_{O_p(n^{-3/2})})}\} - n^{-1}(\beta_1^{(0)*})^2 + O(n^{-2})$$

$$= \beta_2^{(0)} + n^{-1}\beta_{H2}^{(0)} + nE_{T\boldsymbol{\alpha}_0}[\{(\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}] + O(n^{-2})$$

$$= \beta_2^{(0)} + n^{-1}[\beta_{H2}^{(0)} + n^2 E_{T\boldsymbol{\alpha}_0}[\{(\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}]]_{O(1)+O(n^2N^{-1})} + O(n^{-2})$$

$$\equiv \beta_2^{(0)} + n^{-1}(\beta_{H2}^{(0)} + \bar{\beta}_{H2}^{(\Delta)}) + O(n^{-2})$$

$$\equiv \beta_2^{(0)} + n^{-1}(\beta_{H2}^{(0)} + \bar{c}^{**}\beta_{H2}^{(\Delta)}) + O(n^{-2}),$$

where $\bar{\beta}_{H2}^{(\Delta)} = n^2 E_{T\boldsymbol{\alpha}_0}[\{(\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})^2\}_{O_p(N^{-1})}] = \bar{c}^{**}\beta_{H2}^{(\Delta)} > 0,\ \ \bar{\beta}_2 = \beta_2^{(0)},$

$\bar{\beta}_{\text{H}2} = \beta_{\text{H}2}^{(0)} + \bar{\beta}_{\text{H}2}^{(\Delta)} > \beta_{\text{H}2}^{(0)}$, and the underscored term after taking the expectation leaves only the quantity associated with $\beta_{\text{H}2}^{(0)}$.

$$\kappa_3(w) = [\ n^{3/2}[\ \{\text{E}_{\text{T}}\{(q_{O_p(n^{-1/2})}^{(1)})^3\}\}_{O(n^{-2})}$$
$$+3\text{E}_{\text{T}}\{(q_{O_p(n^{-1/2})}^{(1)})^2 q_{O_p(n^{-1})}^{(2)}\}\}_{O(n^{-2})} - 3n^{-2}\beta_1^{(0)*}\beta_2^{(0)}]\ ]_{O(n^{-1/2})} + O(n^{-3/2})$$
$$= n^{-1/2}\beta_3^{(0)} + O(n^{-3/2}),$$

where $\text{E}_{\text{T}}\{\cdot\} = \text{E}_{\text{T}\theta_0}\{\cdot\}$ and $\bar{\beta}_3 = \beta_3^{(0)}\ (\bar{\beta}_3^{(\Delta)} = 0)$.

Using simple notations like $q^{(1)}$,

$$\kappa_4(w) = n^2[\ \text{E}_{\text{T}}\{(q^{(1)})^4 + 4(q^{(1)})^3 q^{(2)} + 6(q^{(1)})^2(q^{(2)})^2 + 4(q^{(1)})^3 q^{(3)}\}$$
$$-3n^{-2}(\beta_2^{(0)})^2 - 4n^{-3}\beta_1^{(0)*}\beta_3^{(0)} - 6n^{-3}\beta_2^{(0)}(\beta_{\text{H}2}^{(0)} + \bar{\beta}_{\text{H}2}^{(\Delta)})$$
$$-6n^{-3}\beta_2^{(0)}(\beta_1^{(0)*})^2\ \ ]_{O(n^{-3})} + O(n^{-2})$$
$$= [n^2[\ [\text{E}_{\text{T}\theta_0}\{(q^{(10)})^4\}]_{O(n^{-2})} + 4[\text{E}_{\text{T}\theta_0}\{(q^{(10)})^3 q^{(20)}\}]_{O(n^{-3})}$$
$$+6[\text{E}_{\text{T}\theta_0}\{(q^{(10)})^2(q^{(20)})^2\}]_{O(n^{-3})} \underline{+6n^{-3}\beta_2^{(0)}\bar{\beta}_{\text{H}2}^{(\Delta)}}$$
$$+4[\text{E}_{\text{T}\theta_0}\{(q^{(10)})^3 q^{(30)}\}]_{O(n^{-3})} - 3n^{-2}(\beta_2^{(0)})^2 - 4n^{-3}\beta_1^{(0)*}\beta_3^{(0)}$$
$$\underline{-6n^{-3}\beta_2^{(0)}}(\beta_{\text{H}2}^{(0)} + \underline{\bar{\beta}_{\text{H}2}^{(\Delta)}}) - 6n^{-3}\beta_2^{(0)}(\beta_1^{(0)*})^2\ \ ]\ ]_{O(n^{-1})} + O(n^{-2})$$
$$= n^{-1}\beta_4^{(0)} + O(n^{-2}),$$

where the sum of the underscored terms is zero, and $\bar{\beta}_4 = \beta_4^{(0)}\ (\bar{\beta}_4^{(\Delta)} = 0)$.

A.4 Asymptotic cumulants of the studentized $\hat{\theta}$: Proof of Theorem 4 under Condition A: $N = O(n)\ (\bar{c} = n/N = O(1))$

Define the asymptotic cumulants $\bar{\beta}_{tk}(k = 1,\ldots,4)$ and the higher-order asymptotic variance $\bar{\beta}_{t\text{H}2}$ with the associated quantities for *t*, which are independent of *n*, in the following equations.

$$\kappa_1(t) = n^{-1/2}\{\mathrm{E}_{\mathrm{T}}(nt^{(2)}_{O_p(n^{-1})}) - \lambda_{\theta_0}^{-1}\eta_{\theta_0}\overline{\beta}_{2I}^{-1/2})\} + O(n^{-3/2})$$

$$= n^{-1/2}\{\overline{\beta}_1\overline{\beta}_{2I}^{-1/2} + \mathrm{E}_{\mathrm{T}}(nq^{(1)}_{O_p(n^{-1/2})}b^{(1)}_{O_p(n^{-1/2})})\} + O(n^{-3/2})$$

$$= n^{-1/2}\{\overline{\beta}_1\overline{\beta}_{2I}^{-1/2} + \mathrm{E}_{\mathrm{T}\theta_0}(nq^{(10)}_{O_p(n^{-1/2})}b^{(10)}_{O_p(n^{-1/2})})$$

$$+ \overline{c}\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(Nq^{(11)}_{O_p(N^{-1/2})}b^{(11)}_{O_p(N^{-1/2})})\} + O(n^{-3/2})$$

$$\equiv n^{-1/2}(\overline{\beta}_1\overline{\beta}_{2I}^{-1/2} + \beta_1^{(t0)} + \overline{\beta}_1^{(t\Delta)}) + O(n^{-3/2})$$

$$\equiv n^{-1/2}(\overline{\beta}_1\overline{\beta}_{2I}^{-1/2} + \beta_1^{(t0)} + \overline{c}\,\beta_1^{(t\Delta)}) + O(n^{-3/2})$$

$$\equiv n^{-1/2}\overline{\beta}_{t1} + O(n^{-3/2})$$

$$= n^{-1/2}(\overline{\beta}_1\overline{\beta}_{2I}^{-1/2} + \beta_1^{(t0)}) + N^{-1/2}\overline{c}^{1/2}\beta_1^{(t\Delta)} + O(n^{-3/2}),$$

where the term $-n^{-1/2}\lambda_{\theta_0}^{-1}\eta_{\theta_0}\overline{\beta}_{2I}^{-1/2}$ is included in $n^{-1/2}\overline{\beta}_1\overline{\beta}_{2I}^{-1/2}$.

Let $\overline{\beta}_{t1}^{*} = \overline{\beta}_{t1} + \lambda_{\theta_0}^{-1}\eta_{\theta_0}\overline{\beta}_{2I}^{-1/2}$. Then,

$$\kappa_2(t) = \mathrm{E}_{\mathrm{T}}\{n(q^{(1)}_{O_p(n^{-1/2})})^2\}\overline{\beta}_{2I}^{-1} + n\mathrm{E}_{\mathrm{T}}[\{(t^{(2)}_{O_p(n^{-1})})^2$$

$$+ 2t^{(1)}_{O_p(n^{-1/2})}(t^{(2)}_{O_p(n^{-1})} + t^{(3)}_{O_p(n^{-3/2})})\}]_{O(n^{-2})} - n^{-1}(\overline{\beta}_{t1}^{*})^2 + O(n^{-2})$$

$$= \overline{\beta}_2\overline{\beta}_{2I}^{-1} + n[\mathrm{E}_{\mathrm{T}}\{(q^{(2)}_{O_p(n^{-1})})^2 + 2q^{(1)}_{O_p(n^{-1/2})}(q^{(2)}_{O_p(n^{-1})} + q^{(3)}_{O_p(n^{-3/2})})\}\overline{\beta}_{2I}^{-1}$$

$$- n^{-2}(\overline{\beta}_1^{*}\overline{\beta}_{2I}^{-1/2})^2]_{O(n^{-2})} + O(n^{-2}) \text{ (recall that } \overline{\beta}_1^{*} = \overline{\beta}_1 + \lambda_{\theta_0}^{-1}\eta_{\theta_0})$$

$$+n\underset{\text{(A)}}{[}\ \mathrm{E}_{\mathrm{T}\theta_0}\{(q^{(10)}_{O_p(n^{-1/2})}b^{(10)}_{O_p(n^{-1/2})})^2 + 2q^{(10)}_{O_p(n^{-1/2})}b^{(10)}_{O_p(n^{-1/2})}q^{(20)}_{O_p(n^{-1})}\overline{\beta}_{2I}^{-1/2}\}$$

$$+ \mathrm{E}_{\mathrm{T}}\{(q^{(10)}_{O_p(n^{-1/2})}b^{(11)}_{O_p(N^{-1/2})})^2 + (q^{(11)}_{O_p(N^{-1/2})}b^{(10)}_{O_p(n^{-1/2})})^2$$

$$+ 4q^{(10)}_{O_p(n^{-1/2})}b^{(10)}_{O_p(n^{-1/2})}q^{(11)}_{O_p(N^{-1/2})}b^{(11)}_{O_p(N^{-1/2})}$$

$$+ 2(\ (q^{(10)}_{O_p(n^{-1/2})}b^{(11)}_{O_p(N^{-1/2})} + q^{(11)}_{O_p(N^{-1/2})}b^{(10)}_{O_p(n^{-1/2})})q^{(21)}_{O_p(n^{-1/2}N^{-1/2})}$$

$$+ q^{(10)}_{O_p(n^{-1/2})}b^{(10)}_{O_p(n^{-1/2})}q^{(22)}_{O_p(N^{-1})} + q^{(11)}_{O_p(N^{-1/2})}b^{(11)}_{O_p(N^{-1/2})}q^{(20)}_{O_p(n^{-1})})\overline{\beta}_{2I}^{-1/2}\}$$

$$+\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(11)}_{O_p(N^{-1/2})}b^{(11)}_{O_p(N^{-1/2})})^2 + 2q^{(11)}_{O_p(N^{-1/2})}b^{(11)}_{O_p(N^{-1/2})}q^{(22)}_{O_p(N^{-1})}\overline{\beta}_{2I}^{-1/2}\}$$

$$+2\mathrm{E}_{\mathrm{T}\theta_0}\{(q^{(10)}_{O_p(n^{-1/2})})^2 b^{(10)}_{O_p(n^{-1/2})}\}\bar{\beta}_{2I}^{-1/2} + 2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(11)}_{O_p(N^{-1/2})})^2 b^{(11)}_{O_p(N^{-1/2})}\}\bar{\beta}_{2I}^{-1/2}$$

$$+2\mathrm{E}_{\mathrm{T}\theta_0}\{(q^{(10)}_{O_p(n^{-1/2})})^2 b^{(20)}_{O_p(n^{-1})}\}\bar{\beta}_{2I}^{-1/2} + 2\mathrm{E}_{\mathrm{T}}\{2q^{(10)}_{O_p(n^{-1/2})} q^{(11)}_{O_p(N^{-1/2})} b^{(21)}_{O_p(n^{-1/2}N^{-1/2})}$$

$$+(q^{(10)}_{O_p(n^{-1/2})})^2 b^{(22)}_{O_p(N^{-1})} + (q^{(11)}_{O_p(N^{-1/2})})^2 b^{(20)}_{O_p(n^{-1})}\}\bar{\beta}_{2I}^{-1/2}$$

$$+2n^{-1}(\beta_2^{(0)} + \bar{\beta}_2^{(\Delta)})\bar{\beta}_{2I}^{-1/2}$$

$$\times\left(n^{-1}\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial \bar{\beta}_{2I}}{\partial \theta_0}\lambda_{\theta_0}^{-1}\eta_{\theta_0} + N^{-1}\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial \bar{\beta}_{2I}}{\partial \boldsymbol{\alpha}_0{}'}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\right)$$

$$+2\mathrm{E}_{\mathrm{T}\theta_0}(q^{(10)}_{O_p(n^{-1/2})} b^{(20)}_{O_p(n^{-1})} b^{(10)}_{O_p(n^{-1/2})})\bar{\beta}_{2I}^{-1/2}$$

$$+2\mathrm{E}_{\mathrm{T}}\{q^{(10)}_{O_p(n^{-1/2})}(q^{(21)}_{O_p(n^{-1/2}N^{-1/2})} b^{(11)}_{O_p(N^{-1/2})} + q^{(22)}_{O_p(N^{-1})} b^{(10)}_{O_p(n^{-1/2})})$$

$$+q^{(11)}_{O_p(N^{-1/2})}(q^{(21)}_{O_p(n^{-1/2}N^{-1/2})} b^{(10)}_{O_p(n^{-1/2})} + q^{(20)}_{O_p(n^{-1})} b^{(11)}_{O_p(N^{-1/2})})\}\bar{\beta}_{2I}^{-1/2}$$

$$+2\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(q^{(11)}_{O_p(N^{-1/2})} q^{(22)}_{O_p(N^{-1})} b^{(11)}_{O_p(N^{-1/2})})\bar{\beta}_{2I}^{-1/2} - 2n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0}$$

$$\times\{\mathrm{E}_{\mathrm{T}\theta_0}(q^{(10)}_{O_p(n^{-1/2})} b^{(10)}_{O_p(n^{-1/2})}) + \mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}(q^{(11)}_{O_p(N^{-1/2})} b^{(11)}_{O_p(N^{-1/2})})\}\bar{\beta}_{2I}^{-1/2}\ \ \underset{(\mathrm{A})}{]}_{O(n^{-2})}$$

$$-n^{-1}(\beta_{t1}^{*})^2 + n^{-1}(\bar{\beta}_1^{*}\bar{\beta}_{2I}^{-1/2})^2 + O(n^{-2})$$

$$\equiv \bar{\beta}_2\bar{\beta}_{2I}^{-1} + n^{-1}\bar{\beta}_{\mathrm{H}2}\bar{\beta}_{2I}^{-1} + n^{-1}\bar{\beta}_{\mathrm{H}2}^{(t0\Delta)} + O(n^{-2})$$

$$\equiv \bar{\beta}_2\bar{\beta}_{2I}^{-1} + n^{-1}\bar{\beta}_{t\mathrm{H}2} + O(n^{-2})$$

$$\equiv \bar{\beta}_{t2} + n^{-1}\bar{\beta}_{t\mathrm{H}2} + O(n^{-2}),$$

where $n^{-1}\bar{\beta}_{\mathrm{H}2}^{(t0\Delta)} = n \underset{(\mathrm{A})}{[} \cdot \underset{(\mathrm{A})}{]} - n^{-1}(\beta_{t1}^{*})^2 + n^{-1}(\beta_1^{*}\bar{\beta}_{2I}^{-1/2})^2$.

$$\kappa_3(t) = n^{3/2}[\ \{\mathrm{E}_{\mathrm{T}}\{(t^{(1)}_{O_p(n^{-1/2})})^3 + 3(t^{(1)}_{O_p(n^{-1/2})})^2 t^{(2)}_{O_p(n^{-1})}\}\}_{O(n^{-2})}$$

$$-3n^{-2}\bar{\beta}_{t1}^{*}\bar{\beta}_{t2}]_{O(n^{-1/2})} + O(n^{-3/2})$$

$$= n^{3/2}[\mathrm{E}_{\mathrm{T}}\{(q^{(1)}_{O_p(n^{-1/2})})^3 + 3(q^{(1)}_{O_p(n^{-1/2})})^2 q^{(2)}_{O_p(n^{-1})}\} - 3n^{-2}\bar{\beta}_1^{*}\bar{\beta}_2]_{O(n^{-2})}\bar{\beta}_{2I}^{-3/2}$$

$$\left(\begin{array}{l}\text{note that } \ \overline{\beta}_1^* = \overline{\beta}_1 + \lambda_{\theta_0}^{-1}\eta_{\theta_0} \ \text{ and} \\ \overline{\beta}_{t1}^* = \overline{\beta}_{t1} + \lambda_{\theta_0}^{-1}\eta_{\theta_0}\overline{\beta}_{2I}^{-1/2} = \overline{\beta}_1\overline{\beta}_{2I}^{-1/2} + \beta_1^{(t0)} + \overline{\beta}_1^{(t\Delta)} + \lambda_{\theta_0}^{-1}\eta_{\theta_0}\overline{\beta}_{2I}^{-1/2} \\ = \overline{\beta}_1^*\overline{\beta}_{2I}^{-1/2} + \beta_1^{(t0)} + \overline{\beta}_1^{(t\Delta)}\end{array}\right)$$

$$+n^{3/2}[\,3\mathrm{E}_{\mathrm{T}\theta_0}\{(q^{(10)}_{O_p(n^{-1/2})})^3 b^{(10)}_{O_p(n^{-1/2})}\overline{\beta}_{2I}^{-1} - 3n^{-2}\beta_1^{(t0)}\overline{\beta}_{t2}\,]_{O(n^{-2})}$$

$$+n^{3/2}[9\mathrm{E}_{\mathrm{T}}\{(q^{(10)}_{O_p(n^{-1/2})})^2 q^{(11)}_{O_p(N^{-1/2})} b^{(11)}_{O_p(N^{-1/2})}$$

$$+(q^{(11)}_{O_p(N^{-1/2})})^2 q^{(10)}_{O_p(n^{-1/2})} b^{(10)}_{O_p(n^{-1/2})}\}\overline{\beta}_{2I}^{-1}$$

$$+3\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{(q^{(11)}_{O_p(N^{-1/2})})^3 b^{(11)}_{O_p(N^{-1/2})}\}\overline{\beta}_{2I}^{-1} - 3n^{-2}\overline{\beta}_1^{(t\Delta)}\overline{\beta}_{t2}\ \,]_{O(n^{-2})} + O(n^{-3/2})$$

$$\equiv n^{-1/2}(\overline{\beta}_3\overline{\beta}_{2I}^{-3/2} + \beta_3^{(t0)} + \overline{\beta}_3^{(t\Delta)}) + O(n^{-3/2})$$

$$\equiv n^{-1/2}\overline{\beta}_{t3} + O(n^{-3/2}).$$

$$\kappa_4(t) = n^2[\ \{\mathrm{E}_{\mathrm{T}}\{(t^{(1)}_{O_p(n^{-1/2})})^4 - 3n^{-2}\overline{\beta}_2^2\overline{\beta}_{2I}^{-2} + 4(t^{(1)}_{O_p(n^{-1/2})})^3 t^{(2)}_{O_p(n^{-1})}$$

$$+6(t^{(1)}_{O_p(n^{-1/2})})^2(t^{(2)}_{O_p(n^{-1})})^2 + 4(t^{(1)}_{O_p(n^{-1/2})})^3 t^{(3)}_{O_p(n^{-3/2})}\}\}_{O(n^{-3})}$$

$$-4n^{-3}\overline{\beta}_{t1}^*\overline{\beta}_{t3} - 6n^{-3}\overline{\beta}_{t2}\overline{\beta}_{t\mathrm{H}2} - 6n^{-3}\overline{\beta}_{t2}(\overline{\beta}_{t1}^*)^2\ \ ]_{O(n^{-3})} + O(n^{-2})$$

$$\left(\text{note that } \ \overline{\beta}_{t3} = \overline{\beta}_3\overline{\beta}_{2I}^{-3/2} + \beta_3^{(t0)} + \overline{\beta}_3^{(t\Delta)} \ \text{ and } \ \overline{\beta}_{t\mathrm{H}2} = \overline{\beta}_{\mathrm{H}2}\overline{\beta}_{2I}^{-1} + \overline{\beta}_{\mathrm{H}2}^{(t0\Delta)}\right)$$

$$= n^2[\ \{\mathrm{E}_{\mathrm{T}}\{(q^{(1)}_{O_p(n^{-1/2})})^4 - 3n^{-2}\overline{\beta}_2^2 + 4(q^{(1)}_{O_p(n^{-1/2})})^3 q^{(2)}_{O_p(n^{-1})}$$

$$+6(q^{(1)}_{O_p(n^{-1/2})})^2(q^{(2)}_{O_p(n^{-1})})^2 + 4(q^{(1)}_{O_p(n^{-1/2})})^3 q^{(3)}_{O_p(n^{-3/2})}\}\}_{O(n^{-3})}$$

$$-4n^{-3}\overline{\beta}_1^*\overline{\beta}_3 - 6n^{-3}\overline{\beta}_2\overline{\beta}_{\mathrm{H}2} - 6n^{-3}\overline{\beta}_2(\overline{\beta}_1^*)^2\ \ ]_{O(n^{-3})}\overline{\beta}_{2I}^{-2} + O(n^{-2})$$

$$+n^2\underset{(\mathrm{A})}{[}\ \mathrm{E}_{\mathrm{T}\theta_0}\{4(q^{(10)}_{O_p(n^{-1/2})})^4 b^{(10)}_{O_p(n^{-1/2})}\overline{\beta}_{2I}^{-3/2} + 6(q^{(10)}_{O_p(n^{-1/2})})^4(b^{(10)}_{O_p(n^{-1/2})})^2\overline{\beta}_{2I}^{-1}$$

$$+12(q^{(10)}_{O_p(n^{-1/2})})^3 b^{(10)}_{O_p(n^{-1/2})} q^{(20)}_{O_p(n^{-1})}\overline{\beta}_{2I}^{-3/2}$$

$$+4(q^{(10)}_{O_p(n^{-1/2})})^4$$

$$\times\left(b^{(20)}_{O_p(n^{-1})} + n^{-1}\frac{\overline{\beta}_{2I}^{-3/2}}{2}\frac{\partial\overline{\beta}_{2I}}{\partial\theta_0}\lambda^{-1}_{\theta_0}\eta_{\theta_0} + N^{-1}\frac{\overline{\beta}_{2I}^{-3/2}}{2}\frac{\partial\overline{\beta}_{2I}}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Lambda}^{-1}_{\boldsymbol{\alpha}_0}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\right)\overline{\beta}_{2I}^{-3/2}$$

$$+4(q^{(10)}_{O_p(n^{-1/2})})^3(q^{(20)}_{O_p(n^{-1})} - n^{-1}\lambda^{-1}_{\theta_0}\eta_{\theta_0})b^{(10)}_{O_p(n^{-1/2})}\overline{\beta}_{2I}^{-3/2} \}$$

$$+\mathrm{E}_{\mathrm{T}}\underset{(\mathrm{B})}{\{} 16(q^{(10)}_{O_p(n^{-1/2})})^3 q^{(11)}_{O_p(N^{-1/2})} b^{(11)}_{O_p(N^{-1/2})}\overline{\beta}_{2I}^{-3/2}$$

$$+24(q^{(10)}_{O_p(n^{-1/2})}q^{(11)}_{O_p(N^{-1/2})})^2(b^{(10)}_{O_p(n^{-1/2})} + b^{(11)}_{O_p(N^{-1/2})})\overline{\beta}_{2I}^{-3/2}$$

$$+16(q^{(11)}_{O_p(N^{-1/2})})^3 q^{(10)}_{O_p(n^{-1/2})} b^{(10)}_{O_p(n^{-1/2})}\overline{\beta}_{2I}^{-3/2}$$

$$+6(q^{(10)}_{O_p(n^{-1/2})})^4(b^{(11)}_{O_p(N^{-1/2})})^2\overline{\beta}_{2I}^{-1} + 6(q^{(11)}_{O_p(N^{-1/2})})^4(b^{(10)}_{O_p(n^{-1/2})})^2\overline{\beta}_{2I}^{-1}$$

$$+36(q^{(10)}_{O_p(n^{-1/2})})^2(q^{(11)}_{O_p(N^{-1/2})})^2\{(b^{(10)}_{O_p(n^{-1/2})})^2 + (b^{(11)}_{O_p(N^{-1/2})})^2\}\overline{\beta}_{2I}^{-1}$$

$$+24q^{(10)}_{O_p(n^{-1/2})}q^{(11)}_{O_p(N^{-1/2})}\{(q^{(10)}_{O_p(n^{-1/2})})^2 + (q^{(11)}_{O_p(N^{-1/2})})^2\}2b^{(10)}_{O_p(n^{-1/2})}b^{(11)}_{O_p(N^{-1/2})}\overline{\beta}_{2I}^{-1}$$

$$+12(q^{(10)}_{O_p(n^{-1/2})})^3(b^{(10)}_{O_p(n^{-1/2})}q^{(22)}_{O_p(N^{-1})} + b^{(11)}_{O_p(N^{-1/2})}q^{(21)}_{O_p(n^{-1/2}N^{-1/2})})\overline{\beta}_{2I}^{-3/2}$$

$$+12(q^{(11)}_{O_p(N^{-1/2})})^3(b^{(11)}_{O_p(N^{-1/2})}q^{(20)}_{O_p(n^{-1})} + b^{(10)}_{O_p(n^{-1/2})}q^{(21)}_{O_p(n^{-1/2}N^{-1/2})})\overline{\beta}_{2I}^{-3/2}$$

$$+36q^{(10)}_{O_p(n^{-1/2})}q^{(11)}_{O_p(N^{-1/2})}(q^{(10)}_{O_p(n^{-1/2})}b^{(10)}_{O_p(n^{-1/2})}q^{(21)}_{O_p(n^{-1/2}N^{-1/2})}$$

$$+q^{(11)}_{O_p(N^{-1/2})}b^{(11)}_{O_p(N^{-1/2})}q^{(21)}_{O_p(n^{-1/2}N^{-1/2})}$$

$$+q^{(10)}_{O_p(n^{-1/2})}b^{(11)}_{O_p(N^{-1/2})}q^{(22)}_{O_p(N^{-1})} + q^{(11)}_{O_p(N^{-1/2})}b^{(10)}_{O_p(n^{-1/2})}q^{(20)}_{O_p(n^{-1})})\overline{\beta}_{2I}^{-3/2}$$

$$+4(q^{(10)}_{O_p(n^{-1/2})})^4 b^{(22)}_{O_p(N^{-1})}\overline{\beta}_{2I}^{-3/2} + 4(q^{(11)}_{O_p(N^{-1/2})})^4 b^{(20)}_{O_p(n^{-1})}\overline{\beta}_{2I}^{-3/2}$$

$$+16q^{(10)}_{O_p(n^{-1/2})}q^{(11)}_{O_p(N^{-1/2})}\{(q^{(10)}_{O_p(n^{-1/2})})^2 + (q^{(11)}_{O_p(N^{-1/2})})^2\}b^{(21)}_{O_p(n^{-1/2}N^{-1/2})}\overline{\beta}_{2I}^{-3/2}$$

$$+24(q^{(10)}_{O_p(n^{-1/2})}q^{(11)}_{O_p(N^{-1/2})})^2\left(b^{(20)}_{O_p(n^{-1})} + b^{(22)}_{O_p(N^{-1})}\right.$$

$$\left. + n^{-1}\frac{\overline{\beta}_{2I}^{-3/2}}{2}\frac{\partial\overline{\beta}_{2I}}{\partial\theta_0}\lambda^{-1}_{\theta_0}\eta_{\theta_0} + N^{-1}\frac{\overline{\beta}_{2I}^{-3/2}}{2}\frac{\partial\overline{\beta}_{2I}}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Lambda}^{-1}_{\boldsymbol{\alpha}_0}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\right)\overline{\beta}_{2I}^{-3/2}$$

$$
\begin{aligned}
&+4(q^{(10)}_{O_p(n^{-1/2})})^3(q^{(22)}_{O_p(N^{-1})}b^{(10)}_{O_p(n^{-1/2})}+q^{(21)}_{O_p(n^{-1/2}N^{-1/2})}b^{(11)}_{O_p(N^{-1/2})})\bar{\beta}_{2I}^{-3/2}\\
&+4(q^{(11)}_{O_p(N^{-1/2})})^3(q^{(20)}_{O_p(n^{-1})}b^{(11)}_{O_p(N^{-1/2})}+q^{(21)}_{O_p(n^{-1/2}N^{-1/2})}b^{(10)}_{O_p(n^{-1/2})})\bar{\beta}_{2I}^{-3/2}\\
&+12(q^{(10)}_{O_p(n^{-1/2})})^2 q^{(11)}_{O_p(N^{-1/2})}\{(q^{(20)}_{O_p(n^{-1})}+q^{(22)}_{O_p(N^{-1})}-n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})b^{(11)}_{O_p(N^{-1/2})}\\
&\qquad\qquad+q^{(21)}_{O_p(n^{-1/2}N^{-1/2})}b^{(10)}_{O_p(n^{-1/2})}\}\bar{\beta}_{2I}^{-3/2}\\
&+12(q^{(11)}_{O_p(N^{-1/2})})^2 q^{(10)}_{O_p(n^{-1/2})}\{(q^{(20)}_{O_p(n^{-1})}+q^{(22)}_{O_p(N^{-1})}-n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})b^{(10)}_{O_p(n^{-1/2})}\\
&\qquad\qquad+q^{(21)}_{O_p(n^{-1/2}N^{-1/2})}b^{(11)}_{O_p(N^{-1/2})}\}\bar{\beta}_{2I}^{-3/2}\qquad\}_{(\mathrm{B})}
\end{aligned}
$$

$$
\begin{aligned}
&+\mathrm{E}_{\mathrm{T}\boldsymbol{\alpha}_0}\{4(q^{(11)}_{O_p(N^{-1/2})})^4 b^{(11)}_{O_p(N^{-1/2})}\bar{\beta}_{2I}^{-3/2}+6(q^{(11)}_{O_p(N^{-1/2})})^4(b^{(11)}_{O_p(N^{-1/2})})^2\bar{\beta}_{2I}^{-1}\\
&\qquad+12(q^{(11)}_{O_p(N^{-1/2})})^3 b^{(11)}_{O_p(N^{-1/2})}q^{(22)}_{O_p(N^{-1})}\bar{\beta}_{2I}^{-3/2}\\
&+4(q^{(11)}_{O_p(N^{-1/2})})^4\\
&\times\left(b^{(22)}_{O_p(N^{-1})}+n^{-1}\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial\theta_0}\lambda_{\theta_0}^{-1}\eta_{\theta_0}+N^{-1}\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial\boldsymbol{\alpha}_0{}'}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\right)\bar{\beta}_{2I}^{-3/2}\\
&\quad+4(q^{(11)}_{O_p(N^{-1/2})})^3(q^{(22)}_{O_p(N^{-1})}-n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})b^{(11)}_{O_p(N^{-1/2})}\bar{\beta}_{2I}^{-3/2}\ \}\\
&-4n^{-3}\{(\beta_1^{(t0)}+\bar{\beta}_1^{(t\Delta)})\bar{\beta}_{t3}+\bar{\beta}_1^{*}\bar{\beta}_{2I}^{-1/2}(\beta_3^{(t0)}+\bar{\beta}_3^{(t\Delta)})\}\\
&-6n^{-3}\bar{\beta}_{\mathrm{H}2}^{(t0\Delta)}\bar{\beta}_{t2}\\
&-6n^{-3}\{(\beta_1^{(t0)}+\bar{\beta}_1^{(t\Delta)})\bar{\beta}_{t1}^{*}\\
&\qquad+\bar{\beta}_1^{*}\bar{\beta}_{2I}^{-1/2}(\beta_1^{(t0)}+\bar{\beta}_1^{(t\Delta)})\}\bar{\beta}_{t2}\qquad]_{(\mathrm{A})O(n^{-3})}+O(n^{-2})\\
&\equiv n^{-1}(\bar{\beta}_4\bar{\beta}_{2I}^{-2}+\bar{\beta}_4^{(t0\Delta)})+O(n^{-2})\\
&\equiv n^{-1}\bar{\beta}_{t4}+O(n^{-2}).
\end{aligned}
$$

A.5 Partial derivatives

A.5.1 Partial derivatives associated with the non-studentized $\hat{\theta}$ under m.m.

Note that under m.m. $P_{\mathrm{T}k}(k=1,...,n)$ are assumed to be not functions of $\boldsymbol{\alpha}$. Define

$$\overline{l}_{\theta_0}(\boldsymbol{\alpha}_0,\theta_0)\equiv n^{-1}\sum_{k=1}^{n}\{U_k\log P_k+(1-U_k)Q_k\}\ .$$

(a.1) $\boldsymbol{\gamma}_{\theta_0}^{(1)}\,'\mathbf{l}_{\theta_0}^{(1)}=\gamma_{\theta_0}^{(1)}l_{\theta_0}^{(1)}=-\lambda_{\theta_0}^{-1}\dfrac{\partial\overline{l}_{\theta_0}}{\partial\theta_0}$

$$\gamma_{\theta_0}^{(\Delta 1)}:\ \frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0}=-\frac{\partial\lambda_{\theta_0}^{-1}}{\partial\boldsymbol{\alpha}_0}=\lambda_{\theta_0}^{-2}\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}=\lambda_{\theta_0}^{-2}\frac{\partial}{\partial\boldsymbol{\alpha}_0}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0^2}\right)$$

$$=\lambda_{\theta_0}^{-2}\frac{\partial}{\partial\boldsymbol{\alpha}_0}n^{-1}\sum_{k=1}^{n}\sum_{P(Q)}^{2}P_{\mathrm{T}k}\left\{-\frac{1}{P_k^2}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2+\frac{1}{P_k}\frac{\partial^2 P_k}{\partial\theta_0^2}\right\}$$

$$=\lambda_{\theta_0}^{-2}n^{-1}\sum_{k=1}^{n}\sum_{P(Q)}^{2}P_{\mathrm{T}k}\left[\left\{\frac{2}{P_k^3}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2-\frac{1}{P_k^2}\frac{\partial^2 P_k}{\partial\theta_0^2}\right\}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\right.$$

$$\left.-\frac{2}{P_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}+\frac{1}{P_k}\frac{\partial^3 P_k}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}\right],$$

where $\sum_{P(Q)}^{2}$ indicates the sum of two terms replacing *P* by *Q* with other summations, shown later, defined similarly. Note that under c.m.s., the above result becomes

$$=\lambda_{\theta_0}^{-2}n^{-1}\sum_{k=1}^{n}\sum_{P(Q)}^{2}\left[\left\{\frac{2}{P_k^2}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2-\frac{1}{P_k}\frac{\partial^2 P_k}{\partial\theta_0^2}\right\}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}-\frac{2}{P_k}\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}\right].$$

On the other hand, since $P_{\mathrm{T}k}(k=1,...,n)$ under c.m.s. are functions of $\boldsymbol{\alpha}$,

$$\lambda_{\theta_0}^{-2}\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}=\lambda_{\theta_0}^{-2}\frac{\partial}{\partial\boldsymbol{\alpha}_0}n^{-1}\sum_{k=1}^{n}\sum_{P(Q)}^{2}\left\{-\frac{1}{P_k}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2\right\}$$

$$=\lambda_{\theta_0}^{-2}n^{-1}\sum_{k=1}^{n}\sum_{P(Q)}^{2}\left\{\frac{1}{P_k^2}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}-\frac{2}{P_k}\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}\right\},$$

which is different from the former result under m.m.

$$\gamma_{\theta_0}^{(\Delta\Delta1)}:\ \frac{\partial^2\gamma_{\theta_0}^{(1)}}{(\partial\boldsymbol{\alpha}_0)^{<2>}}=-2\lambda_{\theta_0}^{-3}\left(\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}\right)^{<2>}+\lambda_{\theta_0}^{-2}\frac{\partial^2\lambda_{\theta_0}}{(\partial\boldsymbol{\alpha}_0)^{<2>}}$$

$$=-2\lambda_{\theta_0}^{-3}\left(\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}\right)^{<2>}$$

$$+\lambda_{\theta_0}^{-2}n^{-1}\sum_{k=1}^{n}\sum_{P(Q)}^{2}P_{\mathrm{T}k}\left[\left\{-\frac{6}{P_k^4}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2+\frac{2}{P_k^3}\frac{\partial^2 P_k}{\partial\theta_0^2}\right\}\left(\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\right)^{<2>}\right.$$

$$+\frac{4}{P_k^3}\frac{\partial P_k}{\partial\theta_0}\sum_{\otimes}^{2}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\otimes\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}-\frac{1}{P_k^2}\sum_{\otimes}^{2}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\otimes\frac{\partial^3 P_k}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}$$

$$+\left\{\frac{2}{P_k^3}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2-\frac{1}{P_k^2}\frac{\partial^2 P_k}{\partial\theta_0^2}\right\}\frac{\partial^2 P_k}{(\partial\boldsymbol{\alpha}_0)^{<2>}}$$

$$\left.-\frac{2}{P_k^2}\left\{\left(\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}\right)^{<2>}+\frac{\partial P_k}{\partial\theta_0}\frac{\partial^3 P_k}{\partial\theta_0(\partial\boldsymbol{\alpha}_0)^{<2>}}\right\}+\frac{1}{P_k}\frac{\partial^4 P_k}{\partial\theta_0^2(\partial\boldsymbol{\alpha}_0)^{<2>}}\right].$$

$$l_{\theta_0}^{(\Delta1)}:\ \frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0}=\frac{\partial}{\partial\boldsymbol{\alpha}_0}n^{-1}\sum_{k=1}^{n}\sum_{\substack{P,U\\(Q,\,1-U)}}^{2}\frac{U_k}{P_k}\frac{\partial P_k}{\partial\theta_0}$$

$$=n^{-1}\sum_{k=1}^{n}\sum_{\substack{P,U\\(Q,\,1-U)}}^{2}U_k\left(-\frac{1}{P_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}+\frac{1}{P_k}\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}\right),$$

$$l_{\theta_0}^{(\Delta\Delta b1)}:\ \frac{\partial^3 \overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0)^{<2>}} = n^{-1}\sum_{k=1}^{n}\sum_{\substack{P,U\\(Q,\,1-U)}}^{2} U_k\left\{\frac{2}{P_k^3}\frac{\partial P_k}{\partial\theta_0}\left(\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\right)^{<2>}\right.$$

$$\left.-\frac{1}{P_k^2}\left(\sum_{\otimes}^{2}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\otimes\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}+\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{(\partial\boldsymbol{\alpha}_0)^{<2>}}\right)+\frac{1}{P_k}\frac{\partial^3 P_k}{\partial\theta_0(\partial\boldsymbol{\alpha}_0)^{<2>}}\right\},$$

$$l_{\theta_0}^{(\Delta\Delta\Delta b1)}:\ \frac{\partial^4 \overline{l}_{\theta_0}}{\partial\theta_0(\partial\boldsymbol{\alpha}_0)^{<3>}} = n^{-1}\sum_{k=1}^{n}\sum_{\substack{P,U\\(Q,\,1-U)}}^{2} U_k\left[-\frac{6}{P_k^4}\frac{\partial P_k}{\partial\theta_0}\left(\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\right)^{<3>}\right.$$

$$+\frac{2}{P_k^3}\sum_{\otimes}^{3}\left\{\left(\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\right)^{<2>}\otimes\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}+\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\otimes\frac{\partial^2 P_k}{(\partial\boldsymbol{\alpha}_0)^{<2>}}\right\}$$

$$-\frac{1}{P_k^2}\left\{\sum_{\otimes}^{3}\left(\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\otimes\frac{\partial^3 P_k}{\partial\theta_0(\partial\boldsymbol{\alpha}_0)^{<2>}}+\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}\otimes\frac{\partial^2 P_k}{(\partial\boldsymbol{\alpha}_0)^{<2>}}\right)\right.$$

$$\left.\left.+\frac{\partial P_k}{\partial\theta_0}\frac{\partial^3 P_k}{(\partial\boldsymbol{\alpha}_0)^{<3>}}\right\}+\frac{1}{P_k}\frac{\partial^4 P_k}{\partial\theta_0(\partial\boldsymbol{\alpha}_0)^{<3>}}\right],$$

where the corresponding expectations are given by replacing U_k with $P_{\mathrm{T}k}(k=1,...,n)$, which hold in the following similar results.

(a.2) $$\boldsymbol{\gamma}_{\theta_0}^{(2)}{}'\mathbf{l}_{\theta_0}^{(2)} = \left\{\lambda_{\theta_0}^{-2}, -\frac{\lambda_{\theta_0}^{-3}}{2}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^3}\right)\right\}\left\{m\frac{\partial\overline{l}_{\theta_0}}{\partial\theta_0}, \left(\frac{\partial\overline{l}_{\theta_0}}{\partial\theta_0}\right)^2\right\}'$$

$$\left(\text{recall } m = \frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0^2} - \lambda_{\theta_0}\right),$$

$$\boldsymbol{\gamma}_{\theta_0}^{(\Delta 2)}:\ \frac{\partial \boldsymbol{\gamma}_{\theta_0}^{(2)}}{\partial \boldsymbol{\alpha}_0{}'}=\left\{-2\lambda_{\theta_0}^{-3},\frac{3\lambda_{\theta_0}^{-4}}{2}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^3}\right)\right\}'\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0{}'}$$

$$+\left\{\mathbf{0},-\frac{\lambda_{\theta_0}^{-3}}{2}\frac{\partial}{\partial\boldsymbol{\alpha}_0}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^3}\right)\right\}',$$

where

$$\frac{\partial}{\partial\boldsymbol{\alpha}_0}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^3}\right)$$

$$=\frac{\partial}{\partial\boldsymbol{\alpha}_0}n^{-1}\sum_{k=1}^{n}\sum_{P(Q)}^{2}P_{\mathrm{T}k}\left\{\frac{2}{P_k^3}\left(\frac{\partial P_k}{\partial\theta_0}\right)^3-\frac{3}{P_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{\partial\theta_0^2}+\frac{1}{P_k}\frac{\partial^3 P_k}{\partial\theta_0^3}\right\}$$

$$=n^{-1}\sum_{k=1}^{n}\sum_{P(Q)}^{2}P_{\mathrm{T}k}\left[\left\{-\frac{6}{P_k^4}\left(\frac{\partial P_k}{\partial\theta_0}\right)^3+\frac{6}{P_k^3}\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{\partial\theta_0^2}-\frac{1}{P_k^2}\frac{\partial^3 P_k}{\partial\theta_0^3}\right\}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\right.$$

$$+\left\{\frac{6}{P_k^3}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2-\frac{3}{P_k^2}\frac{\partial^2 P_k}{\partial\theta_0^2}\right\}\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}$$

$$\left.-\frac{3}{P_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial^3 P_k}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}+\frac{1}{P_k}\frac{\partial^4 P_k}{\partial\theta_0^3\partial\boldsymbol{\alpha}_0}\right].$$

$\mathbf{l}_{\theta_0}^{(\Delta a2)},\ \mathbf{l}_{\theta_0}^{(\Delta b2)},\ \mathbf{l}_{\theta_0}^{(\Delta\Delta a2)},\ \mathbf{l}_{\theta_0}^{(\Delta\Delta b2)},\ \mathbf{l}_{\theta_0}^{(\Delta\Delta c2)}$:

In $\dfrac{\partial m}{\partial\boldsymbol{\alpha}_0}=\dfrac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}-\dfrac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}$, $\dfrac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}$ is given by $\dfrac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}$ shown earlier with $P_{\mathrm{T}k}$ replaced by U_k :

$$\frac{\partial^3\overline{l}_{\theta_0}}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}=\frac{\partial}{\partial\boldsymbol{\alpha}_0}n^{-1}\sum_{k=1}^{n}\sum_{\substack{P,U\\(Q,1-U)}}^{2}U_k\left\{-\frac{1}{P_k^2}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2+\frac{1}{P_k}\frac{\partial^2 P_k}{\partial\theta_0^2}\right\}$$

$$= n^{-1}\sum_{k=1}^{n}\sum_{\substack{P,U\\(Q,1-U)}}^{2} U_k\left[\left\{\frac{2}{P_k^3}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2-\frac{1}{P_k^2}\frac{\partial^2 P_k}{\partial\theta_0^2}\right\}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\right.$$

$$\left.-\frac{2}{P_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}+\frac{1}{P_k}\frac{\partial^3 P_k}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}\right],$$

and in $\dfrac{\partial^2 m}{(\partial\boldsymbol{\alpha}_0)^{<2>}} = \mathrm{E}_{\mathrm{T}\theta_0}\left(\dfrac{\partial^4 \overline{l}_{\theta_0}}{\partial\theta_0^2(\partial\boldsymbol{\alpha}_0)^{<2>}}\right)-\dfrac{\partial^2\lambda_{\theta_0}}{(\partial\boldsymbol{\alpha}_0)^{<2>}}$,

$$\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^4 \overline{l}_{\theta_0}}{\partial\theta_0^2(\partial\boldsymbol{\alpha}_0)^{<2>}}\right)$$

$$= n^{-1}\sum_{k=1}^{n}\sum_{P(Q)}^{2} P_{\mathrm{T}k}\left[\left\{-\frac{6}{P_k^4}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2+\frac{2}{P_k^3}\frac{\partial^2 P_k}{\partial\theta_0^2}\right\}\left(\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\right)^{<2>}\right.$$

$$+\sum_{\otimes}^{2}\left(\frac{4}{P_k^3}\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\otimes\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}-\frac{1}{P_k^2}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\otimes\frac{\partial^3 P_k}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}\right)$$

$$+\left\{\frac{2}{P_k^3}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2-\frac{1}{P_k^2}\frac{\partial^2 P_k}{\partial\theta_0^2}\right\}\frac{\partial^2 P_k}{(\partial\boldsymbol{\alpha}_0)^{<2>}}$$

$$\left.-\frac{2}{P_k^2}\left\{\left(\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}\right)^{<2>}+\frac{\partial P_k}{\partial\theta_0}\frac{\partial^3 P_k}{\partial\theta_0(\partial\boldsymbol{\alpha}_0)^{<2>}}\right\}+\frac{1}{P_k}\frac{\partial^4 P_k}{\partial\theta_0^2(\partial\boldsymbol{\alpha}_0)^{<2>}}\right].$$

(a.3)

$$\boldsymbol{\gamma}_{\theta_0}^{(3)}{}'\mathbf{l}_{\theta_0}^{(3)}$$

$$= \Bigg[-\lambda_{\theta_0}^{-3},\ \frac{3}{2}\lambda_{\theta_0}^{-4}\mathrm{E}_{\mathrm{T}\theta_0}(j_0^{(3)}),\ -\frac{\lambda_{\theta_0}^{-3}}{2},$$

$$-\frac{\lambda_{\theta_0}^{-5}}{2}\{\mathrm{E}_{\mathrm{T}\theta_0}(j_0^{(3)})\}^2 + \frac{\lambda_{\theta_0}^{-4}}{6}\mathrm{E}_{\mathrm{T}\theta_0}(j_0^{(4)}),$$

$$\left\{ \lambda_{\theta_0}^{-2}\eta_{\theta_0},\quad \lambda_{\theta_0}^{-2}\frac{\partial \eta_{\theta_0}}{\partial \theta_0} - \lambda_{\theta_0}^{-3}\mathrm{E}_{\mathrm{T}\theta_0}(j_0^{(3)})\eta_{\theta_0} \right\} \Bigg]$$

$$\times \Bigg[m^2 \frac{\partial \overline{l}_{\theta_0}}{\partial \theta_0},\ m\left(\frac{\partial \overline{l}_{\theta_0}}{\partial \theta_0}\right)^2, \{ j_0^{(3)} - \mathrm{E}_{\mathrm{T}\theta_0}(j_0^{(3)})\}\left(\frac{\partial \overline{l}_{\theta_0}}{\partial \theta_0}\right)^2,$$

$$\left(\frac{\partial \overline{l}_{\theta_0}}{\partial \theta_0}\right)^3,\ n^{-1}\left(m,\ \frac{\partial \overline{l}_{\theta_0}}{\partial \theta_0} \right) \Bigg]'$$

For $\boldsymbol{\gamma}_{\theta_0}^{(3)}$, no partial derivatives are required since $\boldsymbol{\gamma}_{\theta_0}^{(3)}$ is not expended.

$\mathbf{l}_{\theta_0}^{(\Delta a3)},\ \mathbf{l}_{\theta_0}^{(\Delta b3)},\ \mathbf{l}_{\theta_0}^{(\Delta c3)}$:

In $\frac{\partial}{\partial \boldsymbol{\alpha}_0}\left\{ \frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^3} - \mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^3}\right) \right\}$, the following is required and is given by $\frac{\partial}{\partial \boldsymbol{\alpha}_0}\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^3}\right)$ shown earlier with $P_{\mathrm{T}k}$ replaced by U_k :

$$\frac{\partial^4 \overline{l}_{\theta_0}}{\partial\theta_0^3 \partial\boldsymbol{\alpha}} = \frac{\partial}{\partial\boldsymbol{\alpha}_0} n^{-1} \sum_{k=1}^{n} \sum_{\substack{P,U \\ (Q,1-U)}}^{2} U_k \left\{ \frac{2}{P_k^3}\left(\frac{\partial P_k}{\partial\theta_0}\right)^3 - \frac{3}{P_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{\partial\theta_0^2} + \frac{1}{P_k}\frac{\partial^3 P_k}{\partial\theta_0^3} \right\}$$

$$= n^{-1} \sum_{k=1}^{n} \sum_{\substack{P,U \\ (Q,1-U)}}^{2} U_k \left[\left\{ -\frac{6}{P_k^4}\left(\frac{\partial P_k}{\partial\theta_0}\right)^3 + \frac{6}{P_k^3}\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{\partial\theta_0^2} - \frac{1}{P_k^2}\frac{\partial^3 P_k}{\partial\theta_0^3} \right\} \frac{\partial P_k}{\partial\boldsymbol{\alpha}_0} \right.$$

$$\left. + \left\{ \frac{6}{P_k^3}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2 - \frac{3}{P_k^2}\frac{\partial^2 P_k}{\partial\theta_0^2} \right\} \frac{\partial^2 P_k}{\partial\theta_0 \partial\boldsymbol{\alpha}} - \frac{3}{P_k^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial^3 P_k}{\partial\theta_0^2 \partial\boldsymbol{\alpha}} + \frac{1}{P_k}\frac{\partial^4 P_k}{\partial\theta_0^3 \partial\boldsymbol{\alpha}} \right].$$

(a.4) $-n^{-1}(\lambda_{\theta_0}^{-1}\eta_{\theta_0})^{(\Delta)}$

The required $\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0}$ was given earlier while $\frac{\partial\eta_{\theta_0}}{\partial\boldsymbol{\alpha}_0}$ depends on the functional form of η_{θ_0}.

A.5.2 Partial derivatives associated with the non-studentized $\hat{\theta}$ under c.m.s.

Note that under c.m.s. $P_{\mathrm{T}k}(k=1,...,n)$ are functions of $\boldsymbol{\alpha}$. The results different from those in Subsection A.5.1 are only for $\boldsymbol{\gamma}_{\theta_0}^{(k)}(k=1,2)$.

(a.1) $\boldsymbol{\gamma}_{\theta_0}^{(1)}{}'\mathbf{l}_{\theta_0}^{(1)} = \gamma_{\theta_0}^{(1)} l_{\theta_0}^{(1)} = -\lambda_{\theta_0}^{-1}\frac{\partial\overline{l}_{\theta_0}}{\partial\theta_0}$

$$\gamma_{\theta_0}^{(\Delta 1)}: \frac{\partial\gamma_{\theta_0}^{(1)}}{\partial\boldsymbol{\alpha}_0} = -\frac{\partial\lambda_{\theta_0}^{-1}}{\partial\boldsymbol{\alpha}_0} = \lambda_{\theta_0}^{-2}\frac{\partial\lambda_{\theta_0}}{\partial\boldsymbol{\alpha}_0} = \lambda_{\theta_0}^{-2}\frac{\partial}{\partial\boldsymbol{\alpha}_0}\mathrm{E}_{\theta_0}\left(\frac{\partial^2\overline{l}_{\theta_0}}{\partial\theta_0^2}\right)$$

$$= \lambda_{\theta_0}^{-2}\frac{\partial}{\partial\boldsymbol{\alpha}_0} n^{-1}\sum_{k=1}^{n}\sum_{P(Q)}^{2}\left\{-\frac{1}{P_k}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2\right\}$$

$$= \lambda_{\theta_0}^{-2} n^{-1} \sum_{k=1}^{n} \sum_{P(Q)}^{2} \left\{ \frac{1}{P_k^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial P_k}{\partial \boldsymbol{\alpha}_0} - \frac{2}{P_k} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0 \partial \boldsymbol{\alpha}_0} \right\},$$

$$\gamma_{\theta_0}^{(\Delta\Delta 1)}: \ \frac{\partial^2 \gamma_{\theta_0}^{(1)}}{(\partial \boldsymbol{\alpha}_0)^{<2>}} = -2\lambda_{\theta_0}^{-3} \left(\frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right)^{<2>} + \lambda_{\theta_0}^{-2} \frac{\partial^2 \lambda_{\theta_0}}{(\partial \boldsymbol{\alpha}_0)^{<2>}}$$

$$= -2\lambda_{\theta_0}^{-3} \left(\frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0} \right)^{<2>} + \lambda_{\theta_0}^{-2} n^{-1} \sum_{k=1}^{n} \sum_{P(Q)}^{2} \left[-\frac{2}{P_k^3} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \left(\frac{\partial P_k}{\partial \boldsymbol{\alpha}_0} \right)^{<2>} \right.$$

$$+ \frac{2}{P_k^2} \frac{\partial P_k}{\partial \theta_0} \sum_{\otimes}^{2} \frac{\partial P_k}{\partial \boldsymbol{\alpha}_0} \otimes \frac{\partial^2 P_k}{\partial \theta_0 \partial \boldsymbol{\alpha}_0} + \frac{1}{P_k^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial^2 P_k}{(\partial \boldsymbol{\alpha}_0)^{<2>}}$$

$$\left. - \frac{2}{P_k} \left\{ \left(\frac{\partial^2 P_k}{\partial \theta_0 \partial \boldsymbol{\alpha}_0} \right)^{<2>} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0 (\partial \boldsymbol{\alpha}_0)^{<2>}} \right\} \right].$$

(a.2) $$\boldsymbol{\gamma}_{\theta_0}^{(2)}{}' \mathbf{l}_{\theta_0}^{(2)} = \left\{ \lambda_{\theta_0}^{-2}, -\frac{\lambda_{\theta_0}^{-3}}{2} \mathrm{E}_{\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^3} \right) \right\} \left\{ m \frac{\partial \overline{l}_{\theta_0}}{\partial \theta_0}, \left(\frac{\partial \overline{l}_{\theta_0}}{\partial \theta_0} \right)^2 \right\}',$$

$$\boldsymbol{\gamma}_{\theta_0}^{(\Delta 2)}: \frac{\partial \boldsymbol{\gamma}_{\theta_0}^{(2)}}{\partial \boldsymbol{\alpha}_0{}'} = \left\{ -2\lambda_{\theta_0}^{-3}, \frac{3\lambda_{\theta_0}^{-4}}{2} \mathrm{E}_{\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^3} \right) \right\}' \frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0{}'}$$

$$+ \left\{ \mathbf{0}, -\frac{\lambda_{\theta_0}^{-3}}{2} \frac{\partial}{\partial \boldsymbol{\alpha}_0} \mathrm{E}_{\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^3} \right) \right\}',$$

where

$$\frac{\partial}{\partial \boldsymbol{\alpha}_0} \mathrm{E}_{\theta_0} \left(\frac{\partial^3 \overline{l}_{\theta_0}}{\partial \theta_0^3} \right)$$

$$= \frac{\partial}{\partial \boldsymbol{\alpha}_0} n^{-1} \sum_{k=1}^{n} \sum_{P(Q)}^{2} \left\{ \frac{2}{P_k^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 - \frac{3}{P_k} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0^2} \right\}$$

$$= n^{-1} \sum_{k=1}^{n} \sum_{P(Q)}^{2} \left\{ -\frac{4}{P_k^3} \left(\frac{\partial P_k}{\partial \theta_0} \right)^3 \frac{\partial P_k}{\partial \boldsymbol{\alpha}_0} + \frac{6}{P_k^2} \left(\frac{\partial P_k}{\partial \theta_0} \right)^2 \frac{\partial^2 P_k}{\partial \theta_0 \partial \boldsymbol{\alpha}_0} \right.$$

$$\left. + \frac{3}{P_k^2} \frac{\partial P_k}{\partial \theta_0} \frac{\partial^2 P_k}{\partial \theta_0^2} \frac{\partial P_k}{\partial \boldsymbol{\alpha}_0} - \frac{3}{P_k} \left(\frac{\partial^2 P_k}{\partial \theta_0 \partial \boldsymbol{\alpha}_0} \frac{\partial^2 P_k}{\partial \theta_0^2} + \frac{\partial P_k}{\partial \theta_0} \frac{\partial^3 P_k}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_0} \right) \right\}$$

(a.3) $\boldsymbol{\gamma}_{\theta_0}^{(3)}{}' \mathbf{l}_{\theta_0}^{(3)}$

For $\boldsymbol{\gamma}_{\theta_0}^{(3)}$, no partial derivatives are required since $\boldsymbol{\gamma}_{\theta_0}^{(3)}$ is not expended.

(a.4) $-n^{-1}(\lambda_{\theta_0}^{-1}\eta_{\theta_0})^{(\Delta)}$

$\frac{\partial \lambda_{\theta_0}}{\partial \boldsymbol{\alpha}_0}$ was given earlier while $\frac{\partial \eta_{\theta_0}}{\partial \boldsymbol{\alpha}_0}$ depends on the functional form of η_{θ_0}.

A.5.3 Partial derivatives when the 3PLM is employed in Subsections A.5.1 and A.5.2

Define $\boldsymbol{\alpha}_0 = \{\boldsymbol{\alpha}_{0(1)}{}', ..., \boldsymbol{\alpha}_{0(n)}{}'\}'$, $\boldsymbol{\alpha}_{0(k)} = (a_k, b_k, c_k)'$,

$P_k = c_k + \frac{1 - c_k}{1 + \exp\{-Da_k(\theta_0 - b_k)\}} = c_k + (1 - c_k)B_k \ \ (k = 1, ..., n)$ and $D = 1.7$. Then, assuming k = 1,…,n when unspecified from now on,

$$\frac{\partial P_k}{\partial \boldsymbol{\alpha}_{0(k)}} = \{(\theta_0 - b_k, -a_k) D(1 - c_k) B_k (1 - B_k),\ 1 - B_k\}',$$

$$\frac{\partial^2 P_k}{\partial \theta_0 \partial \boldsymbol{\alpha}_{0(k)}} = [\{D + (\theta_0 - b_k) D^2 a_k (1 - 2B_k), -D^2 a_k^2 (1 - 2B_k)\}(1 - c_k),$$

$$-Da_k]' B_k (1 - B_k),$$

$$\frac{\partial^2 P_k}{\partial \boldsymbol{\alpha}_{0(k)} \partial \boldsymbol{\alpha}_{0(k)}'}$$

$$= \begin{bmatrix} \begin{pmatrix} (\theta_0 - b_k)^2 & -(\theta_0 - b_k) a_k \\ -(\theta_0 - b_k) a_k & a_k^2 \end{pmatrix} D^2 (1 - c_k)(1 - 2B_k) & \\ \quad + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} D(1 - c_k) & \text{sym.} \\ \{-(\theta_0 - b_k),\ a_k\} D & 0 \end{bmatrix} B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_{0(k)}}$$

$$= \left[\left\{ \begin{matrix} D^2 a_k (1 - 2B_k) \\ +(\theta_0 - b_k) D^3 a_k^2 (1 - 6B_k + 6B_k^2), \end{matrix} \quad -D^3 a_k^3 (1 - 6B_k + 6B_k^2) \right\} (1 - c_k), \right.$$

$$\left. - D^2 a_k^2 (1 - 2B_k) \right]' B_k (1 - B_k).$$

$$\frac{\partial^3 P_k}{\partial \theta_0 \partial \boldsymbol{\alpha}_{0(k)} \partial \boldsymbol{\alpha}_{0(k)}'}:$$

$$\frac{\partial^3 P_k}{\partial \theta_0 \partial a_k^2} = \{\ 2(\theta_0 - b_k) D^2 (1 - c_k)(1 - 2B_k) + (\theta_0 - b_k)^2 a_k D^3 (1 - c_k) \times (1 - 6B_k + 6B_k^2)\ \} B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial \theta_0 \partial b_k \partial a_k} = \{\ -2 a_k D^2 (1 - c_k)(1 - 2B_k) - (\theta_0 - b_k) a_k^2 D^3 (1 - c_k) \times (1 - 6B_k + 6B_k^2)\ \} B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial \theta_0 \partial b_k^2} = a_k^3 D^3 (1 - c_k)(1 - 6B_k + 6B_k^2) B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial \theta_0 \partial c_k \partial a_k} = -\{D + (\theta_0 - b_k) a_k D^2 (1 - 2B_k)\} B_k (1 - B_k),$$

$$\frac{\partial^3 P_k}{\partial\theta_0 \partial c_k \partial b_k} = a_k^2 D^2 (1-2B_k) B_k (1-B_k),$$

$$\frac{\partial^3 P_k}{\partial\theta_0 \partial c_k^2} = 0.$$

$$\frac{\partial^3 P_k}{\partial a_k \partial \boldsymbol{\alpha}_{0(k)} \partial \boldsymbol{\alpha}_{0(k)}'}:$$

$$\frac{\partial^3 P_k}{\partial a_k^3} = (\theta_0 - b_k)^3 D^3 (1-c_k)(1-6B_k+6B_k^2) B_k (1-B_k),$$

$$\frac{\partial^3 P_k}{\partial a_k \partial b_k \partial a_k} = \{ \; -2(\theta_0 - b_k) D^2 (1-c_k)(1-2B_k)$$
$$-(\theta_0 - b_k)^2 a_k D^3 (1-c_k)(1-6B_k+6B_k^2) \; \} B_k (1-B_k),$$

$$\frac{\partial^3 P_k}{\partial a_k \partial b_k^2} = \{ \; 2a_k D^2 (1-c_k)(1-2B_k)$$
$$+ a_k^2 D^3 (\theta_0 - b_k)(1-c_k)(1-6B_k+6B_k^2) \; \} B_k (1-B_k),$$

$$\frac{\partial^3 P_k}{\partial a_k \partial c_k \partial a_k} = -(\theta_k - b_k)^2 D^2 (1-2B_k) B_k (1-B_k),$$

$$\frac{\partial^3 P_k}{\partial a_k \partial c_k \partial b_k} = \{ D + (\theta_k - b_k) a_k D^2 (1-2B_k) \} B_k (1-B_k),$$

$$\frac{\partial^3 P_k}{\partial a_k \partial c_k^2} = 0.$$

$$\frac{\partial^3 P_k}{\partial b_k \partial \boldsymbol{\alpha}_{0(k)} \partial \boldsymbol{\alpha}_{0(k)}'}:$$

$$\frac{\partial^3 P_k}{\partial b_k \partial a_k^2} = \{ \; -2(\theta_0 - b_k) D^2 (1-c_k)(1-2B_k)$$
$$-(\theta_0 - b_k)^2 a_k D^3 (1-c_k)(1-6B_k+6B_k^2) \; \} B_k (1-B_k),$$

$$\frac{\partial^3 P_k}{\partial b_k^2 \partial a_k} = \{\ 2a_k D^2 (1-c_k)(1-2B_k)$$

$$+(\theta_0 - b_k)a_k^2 D^3 (1-c_k)(1-6B_k+6B_k^2)\ \} B_k (1-B_k),$$

$$\frac{\partial^3 P_k}{\partial b_k^3} = -a_k^3 D^3 (1-c_k)(1-6B_k+6B_k^2) B_k (1-B_k),$$

$$\frac{\partial^3 P_k}{\partial b_k \partial c_k \partial a_k} = \{D + (\theta_0 - b_k)a_k D^2 (1-2B_k)\} B_k (1-B_k),$$

$$\frac{\partial^3 P_k}{\partial b_k \partial c_k \partial b_k} = -a_k^2 D^2 (1-2B_k) B_k (1-B_k),$$

$$\frac{\partial^3 P_k}{\partial b_k \partial c_k^2} = 0.$$

$$\frac{\partial^3 P_k}{\partial c_k \partial \boldsymbol{\alpha}_{0(k)} \partial \boldsymbol{\alpha}_{0(k)}'}$$

$$= \begin{bmatrix} \begin{pmatrix} -(\theta_0 - b_k)^2 & (\theta_0 - b_k)a_k \\ (\theta_0 - b_k)a_k & -a_k^2 \end{pmatrix} D^2 (1-2B_k) & \begin{matrix} 0 \\ 0 \end{matrix} \\ +\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D & \\ 0 \qquad\qquad 0 & 0 \end{bmatrix} B_k (1-B_k),$$

$$\frac{\partial^4 P_k}{\partial\theta_0^3 \partial \boldsymbol{\alpha}_{0(k)}}$$

$$= \left[\left\{ \begin{array}{ll} 2D^3 a_k^2 (1-6B_k+6B_k^2) & -D^4 a_k^4 (1-14B_k \\ +(\theta_0-b_k) D^4 a_k^3 & \quad +36B_k^2-24B_k^3) \\ \times(1-14B_k+36B_k^2-24B_k^3), & \end{array} \right\} (1-c_k), \right.$$

$$\left. -D^3 a_k^3 (1-6B_k+6B_k^2) \right]' B_k(1-B_k).$$

$$\frac{\partial^4 P_k}{\partial\theta_0^2 \partial \boldsymbol{\alpha}_{0(k)} \partial \boldsymbol{\alpha}_{0(k)}'}:$$

$$\frac{\partial^4 P_k}{\partial\theta_0^2 \partial a_k^2} = \{ \ 2D^2(1-c_k)(1-2B_k)$$
$$+4(\theta_0-b_k)a_k D^3 (1-c_k)(1-6B_k+6B_k^2)$$
$$+(\theta_0-b_k)^2 a_k^2 D^4 (1-c_k)(1-14B_k+36B_k^2-24B_k^3) \ \} B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial\theta_0^2 \partial b_k \partial a_k} = \{ \ -3a_k^2 D^3 (1-c_k)(1-6B_k+6B_k^2)$$
$$-(\theta_0-b_k) a_k^3 D^4 (1-c_k)(1-14B_k+36B_k^2-24B_k^3) \ \} B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial\theta_0^2 \partial b_k^2} = a_k^4 D^4 (1-c_k)(1-14B_k+36B_k^2-24B_k^3) B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial\theta_0^2 \partial c_k \partial a_k} = \{ \ -2a_k D^2 (1-2B_k)$$
$$-(\theta_0-b_k) a_k^2 D^3 (1-6B_k+6B_k^2) \ \} B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0^2 \partial c_k \partial b_k} = a_k^3 D^3 (1 - 6B_k + 6B_k^2) B_k (1 - B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0^2 \partial c_k^2} = 0.$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial a_k \partial \boldsymbol{\alpha}_{0(k)} \partial \boldsymbol{\alpha}_{0(k)}'}:$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial a_k^3} = \{ \quad 3(\theta_0 - b_k)^2 D^3 (1 - c_k)(1 - 6B_k + 6B_k^2)$$
$$+ (\theta_0 - b_k)^3 a_k D^4 (1 - c_k)(1 - 14B_k + 36B_k^2 - 24B_k^3) \quad \} B_k (1 - B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial a_k \partial b_k \partial a_k} = \{ \quad -2D^2 (1 - c_k)(1 - 2B_k)$$
$$- 4(\theta_0 - b_k) a_k D^3 (1 - c_k)(1 - 6B_k + 6B_k^2)$$
$$- (\theta_0 - b_k)^2 a_k^2 D^4 (1 - c_k)(1 - 14B_k + 36B_k^2 - 24B_k^3) \quad \} B_k (1 - B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial a_k \partial b_k^2} = \{ \quad 3a_k^2 D^3 (1 - c_k)(1 - 6B_k + 6B_k^2)$$
$$+ (\theta_0 - b_k) a_k^3 D^4 (1 - c_k)(1 - 14B_k + 36B_k^2 - 24B_k^3) \quad \} B_k (1 - B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial a_k \partial c_k \partial a_k} = \{ \quad -2(\theta_0 - b_k) D^2 (1 - 2B_k)$$
$$- (\theta_0 - b_k)^2 a_k D^3 (1 - 6B_k + 6B_k^2) \quad \} B_k (1 - B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial a_k \partial c_k \partial b_k} = \{ \quad 2a_k D^2 (1 - 2B_k)$$
$$+ (\theta_0 - b_k) a_k^2 D^3 (1 - 6B_k + 6B_k^2) \quad \} B_k (1 - B_k),$$

$$\frac{\partial^4 P_k}{\partial \theta_0 \partial a_k \partial c_k^2} = 0.$$

$$\frac{\partial^4 P_k}{\partial\theta_0 \partial b_k \partial \boldsymbol{\alpha}_{0(k)} \partial \boldsymbol{\alpha}_{0(k)}{}'}:$$

$$\frac{\partial^4 P_k}{\partial\theta_0 \partial b_k \partial a_k^2} = \{ \ -2D^2(1-c_k)(1-2B_k)$$

$$-4(\theta_0 - b_k)a_k D^3(1-c_k)(1-6B_k+6B_k^2)$$

$$-(\theta_0 - b_k)^2 a_k^2 D^4(1-c_k)(1-14B_k+36B_k^2-24B_k^3) \quad \}B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial\theta_0 \partial b_k^2 \partial a_k} = \{ \ 3a_k^2 D^3(1-c_k)(1-6B_k+6B_k^2)$$

$$+(\theta_0 - b_k)a_k^3 D^4(1-c_k)(1-14B_k+36B_k^2-24B_k^3) \quad \}B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial\theta_0 \partial b_k^3} = -a_k^4 D^4(1-c_k)(1-14B_k+36B_k^2-24B_k^3)B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial\theta_0 \partial b_k \partial c_k \partial a_k} = \{ \ 2a_k D^2(1-2B_k)$$

$$+(\theta_0 - b_k)a_k^2 D^3(1-6B_k+6B_k^2) \ \}B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial\theta_0 \partial b_k \partial c_k \partial b_k} = -a_k^3 D^3(1-6B_k+6B_k^2)B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial\theta_0 \partial b_k \partial c_k^2} = 0.$$

$$\frac{\partial^4 P_k}{\partial\theta_0 \partial c_k \partial \boldsymbol{\alpha}_{0(k)} \partial \boldsymbol{\alpha}_{0(k)}{}'}:$$

$$\frac{\partial^4 P_k}{\partial\theta_0 \partial c_k \partial a_k^2} = \{ \ -2(\theta_0 - b_k)D^2(1-2B_k)$$

$$-(\theta_0 - b_k)^2 a_k D^3(1-6B_k+6B_k^2) \quad \}B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial\theta_0 \partial c_k \partial b_k \partial a_k} = \{ \ 2a_k D^2(1-2B_k)$$

$$+(\theta_0 - b_k)a_k^2 D^3(1-6B_k+6B_k^2) \quad \}B_k(1-B_k),$$

$$\frac{\partial^4 P_k}{\partial\theta_0 \partial c_k \partial b_k^2} = -a_k^3 D^3 (1-6B_k + 6B_k^2) B_k (1-B_k),$$

$$\frac{\partial^4 P_k}{\partial\theta_0 \partial c_k^2 \partial a_k} = \frac{\partial^4 P_k}{\partial\theta_0 \partial c_k^2 \partial b_k} = \frac{\partial^4 P_k}{\partial\theta_0 \partial c_k^3} = 0.$$

A.5.4 Partial derivatives associated with the studentized $\hat{\theta}$

(a.1) $i_{\theta_0}^{(1)} = \frac{\overline{i}_{\theta_0}^{-1/2}}{2} \overline{i}_{\theta_0}^{(\mathrm{D1})}$

$$\frac{\partial i_{\theta_0}^{(1)}}{\partial\theta_0} = -\frac{\overline{i}_{\theta_0}^{-3/2}}{4} (\overline{i}_{\theta_0}^{(\mathrm{D1})})^2 + \frac{\overline{i}_{\theta_0}^{-1/2}}{2} \overline{i}_{\theta_0}^{(\mathrm{D2})},$$

where $\overline{i}_{\theta_0}^{(\mathrm{D1})} = n^{-1} \sum_{k=1}^{n} \left\{ \frac{2}{P_k Q_k} \frac{\partial P_k}{\partial\theta_0} \frac{\partial^2 P_k}{\partial\theta_0^2} - \frac{1-2P_k}{(P_k Q_k)^2} \left(\frac{\partial P_k}{\partial\theta_0} \right)^3 \right\}$ and

$$\overline{i}_{\theta_0}^{(\mathrm{D2})} = n^{-1} \sum_{k=1}^{n} \left[\frac{2}{P_k Q_k} \left\{ \frac{\partial P_k}{\partial\theta_0} \frac{\partial^3 P_k}{\partial\theta_0^3} + \left(\frac{\partial^2 P_k}{\partial\theta_0^2} \right)^2 \right\} + \frac{1}{(P_k Q_k)^2} \right.$$

$$\left. \times \left\{ -5(1-2P_k) \left(\frac{\partial P_k}{\partial\theta_0} \right)^2 \frac{\partial^2 P_k}{\partial\theta_0^2} + 2 \left(\frac{\partial P_k}{\partial\theta_0} \right)^4 \right\} + \frac{2(1-2P_k)^2}{(P_k Q_k)^3} \left(\frac{\partial P_k}{\partial\theta_0} \right)^4 \right].$$

$$\frac{\partial i_{\theta_0}^{(1)}}{\partial \boldsymbol{\alpha}_{0(k)}} = -\frac{\overline{i}_{\theta_0}^{-3/2}}{4} \overline{i}_{\theta_0}^{(\mathrm{D1})} \frac{\partial \overline{i}_{\theta_0}}{\partial \boldsymbol{\alpha}_{0(k)}} + \frac{\overline{i}_{\theta_0}^{-1/2}}{2} \frac{\partial \overline{i}_{\theta_0}^{(\mathrm{D1})}}{\partial \boldsymbol{\alpha}_{0(k)}},$$

$$\frac{\partial^2 i_{\theta_0}^{(1)}}{(\partial \boldsymbol{\alpha}_{0(k)})^{<2>}} = \frac{3}{8} \overline{i}_{\theta_0}^{-5/2} \overline{i}_{\theta_0}^{(\mathrm{D1})} \left(\frac{\partial \overline{i}_{\theta_0}}{\partial \boldsymbol{\alpha}_{0(k)}} \right)^{<2>} - \frac{1}{4} \sum_{\otimes}^{2} \overline{i}_{\theta_0}^{-3/2} \frac{\partial \overline{i}_{\theta_0}}{\partial \boldsymbol{\alpha}_{0(k)}} \otimes \frac{\partial \overline{i}_{\theta_0}^{(\mathrm{D1})}}{\partial \boldsymbol{\alpha}_{0(k)}}$$

$$- \frac{1}{4} \overline{i}_{\theta_0}^{-3/2} \overline{i}_{\theta_0}^{(\mathrm{D1})} \frac{\partial^2 \overline{i}_{\theta_0}}{(\partial \boldsymbol{\alpha}_{0(k)})^{<2>}} + \frac{\overline{i}_{\theta_0}^{-1/2}}{2} \frac{\partial^2 \overline{i}_{\theta_0}^{(\mathrm{D1})}}{(\partial \boldsymbol{\alpha}_{0(k)})^{<2>}},$$

$$\frac{\partial^2 i_{\theta_0}^{(1)}}{\partial\theta_0\partial\boldsymbol{\alpha}_{0(k)}} = \frac{3}{8}\overline{i}_{\theta_0}^{-5/2}(\overline{i}_{\theta_0}^{(\mathrm{D1})})^2\frac{\partial\overline{i}_{\theta_0}}{\partial\boldsymbol{\alpha}_{0(k)}} - \frac{\overline{i}_{\theta_0}^{-3/2}}{2}\overline{i}_{\theta_0}^{(\mathrm{D1})}\frac{\partial\overline{i}_{\theta_0}^{(\mathrm{D1})}}{\partial\boldsymbol{\alpha}_{0(k)}}$$

$$-\frac{\overline{i}_{\theta_0}^{-3/2}}{4}\overline{i}_{\theta_0}^{(\mathrm{D2})}\frac{\partial\overline{i}_{\theta_0}}{\partial\boldsymbol{\alpha}_{0(k)}} + \frac{\overline{i}_{\theta_0}^{-1/2}}{2}\frac{\partial\overline{i}_{\theta_0}^{(\mathrm{D2})}}{\partial\boldsymbol{\alpha}_{0(k)}},$$

$$\frac{\partial^2 i_{\theta_0}^{(1)}}{\partial\theta_0^2} = \frac{3}{8}\overline{i}_{\theta_0}^{-5/2}(\overline{i}_{\theta_0}^{(\mathrm{D1})})^3 - \frac{3}{4}\overline{i}_{\theta_0}^{-3/2}\overline{i}_{\theta_0}^{(\mathrm{D1})}\overline{i}_{\theta_0}^{(\mathrm{D2})} + \frac{\overline{i}_{\theta_0}^{-1/2}}{2}\overline{i}_{\theta_0}^{(\mathrm{D3})},$$

where

$$\frac{\partial\overline{i}_{\theta_0}}{\partial\boldsymbol{\alpha}_{0(k)}} = n^{-1}\frac{\partial}{\partial\boldsymbol{\alpha}_{0(k)}}\sum_{k=1}^{n}\frac{1}{P_kQ_k}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2$$

$$= n^{-1}\left\{\frac{2}{P_kQ_k}\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_{0(k)}} - \left(\frac{\partial P_k}{\partial\theta_0}\right)^2\frac{1-2P_k}{(P_kQ_k)^2}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_{0(k)}}\right\},$$

$$\frac{\partial^2\overline{i}_{\theta_0}}{(\partial\boldsymbol{\alpha}_{0(k)})^{<2>}} = n^{-1}\left[\frac{2}{P_kQ_k}\left\{\left(\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_{0(k)}}\right)^{<2>} + \frac{\partial P_k}{\partial\theta_0}\frac{\partial^3 P_k}{\partial\theta_0(\partial\boldsymbol{\alpha}_{0(k)})^{<2>}}\right\}\right.$$

$$+\frac{1}{(P_kQ_k)^2}\left\{-\sum_{\otimes}^{2}2\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_{0(k)}}\otimes\left((1-2P_k)\frac{\partial P_k}{\partial\boldsymbol{\alpha}_{0(k)}}\right)\right.$$

$$\left.+2\left(\frac{\partial P_k}{\partial\theta_0}\right)^2\left(\frac{\partial P_k}{\partial\boldsymbol{\alpha}_{0(k)}}\right)^{<2>} - \left(\frac{\partial P_k}{\partial\theta_0}\right)^2(1-2P_k)\frac{\partial^2 P_k}{(\partial\boldsymbol{\alpha}_{0(k)})^{<2>}}\right\}$$

$$\left.+2\left(\frac{\partial P_k}{\partial\theta_0}\right)^2\frac{(1-2P_k)^2}{(P_kQ_k)^3}\left(\frac{\partial P_k}{\partial\boldsymbol{\alpha}_{0(k)}}\right)^{<2>}\right],$$

$$\overline{i}_{\theta_0}^{(\mathrm{D3})} = n^{-1}\sum_{k=1}^{n}\left[\frac{2}{P_kQ_k}\left(3\frac{\partial^2 P_k}{\partial\theta_0^2}\frac{\partial^3 P_k}{\partial\theta_0^3} + \frac{\partial P_k}{\partial\theta_0}\frac{\partial^4 P_k}{\partial\theta_0^4}\right)\right.$$

$$+\frac{1}{(P_kQ_k)^2}\left\{(1-2P_k)\left\{-2\left(\frac{\partial P_k}{\partial\theta_0}\frac{\partial^3 P_k}{\partial\theta_0^3}+\left(\frac{\partial^2 P_k}{\partial\theta_0^2}\right)^2\right)\frac{\partial P_k}{\partial\theta_0}\right.\right.$$

$$\left.-5\left(2\frac{\partial P_k}{\partial\theta_0}\left(\frac{\partial^2 P_k}{\partial\theta_0^2}\right)^2+\left(\frac{\partial P_k}{\partial\theta_0}\right)^2\frac{\partial^3 P_k}{\partial\theta_0^3}\right)\right\}+18\left(\frac{\partial P_k}{\partial\theta_0}\right)^3\frac{\partial^2 P_k}{\partial\theta_0^2}\right\}$$

$$+\frac{1}{(P_kQ_k)^3}\left\{18(1-2P_k)^2\left(\frac{\partial P_k}{\partial\theta_0}\right)^3\frac{\partial^2 P_k}{\partial\theta_0^2}-12(1-2P_k)\left(\frac{\partial P_k}{\partial\theta_0}\right)^5\right\}$$

$$\left.-\frac{6(1-2P_k)^3}{(P_kQ_k)^4}\left(\frac{\partial P_k}{\partial\theta_0}\right)^5\right],$$

$$\frac{\partial\overline{i}_{\theta_0}^{(\mathrm{D1})}}{\partial\boldsymbol{\alpha}_{0(k)}}=n^{-1}\frac{\partial}{\partial\boldsymbol{\alpha}_{0(k)}}\sum_{k=1}^{n}\left\{\frac{2}{P_kQ_k}\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{\partial\theta_0^2}-\frac{1-2P_k}{(P_kQ_k)^2}\left(\frac{\partial P_k}{\partial\theta_0}\right)^3\right\}$$

$$=n^{-1}\left[\frac{2}{P_kQ_k}\left(\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_{0(k)}}\frac{\partial^2 P_k}{\partial\theta_0^2}+\frac{\partial P_k}{\partial\theta_0}\frac{\partial^3 P_k}{\partial\theta_0^2\partial\boldsymbol{\alpha}_{0(k)}}\right)\right.$$

$$+\frac{1}{(P_kQ_k)^2}\left\{-(1-2P_k)\left(2\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{\partial\theta_0^2}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_{0(k)}}+3\left(\frac{\partial P_k}{\partial\theta_0}\right)^2\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_{0(k)}}\right)\right.$$

$$\left.\left.+2\left(\frac{\partial P_k}{\partial\theta_0}\right)^3\frac{\partial P_k}{\partial\boldsymbol{\alpha}_{0(k)}}\right\}+\frac{2(1-2P_k)^2}{(P_kQ_k)^3}\left(\frac{\partial P_k}{\partial\theta_0}\right)^3\frac{\partial P_k}{\partial\boldsymbol{\alpha}_{0(k)}}\right],$$

$$\frac{\partial^2\overline{i}_{\theta_0}^{(\mathrm{D1})}}{(\partial\boldsymbol{\alpha}_{0(k)})^{<2>}}=n^{-1}\left[\frac{2}{P_kQ_k}\left(\frac{\partial^3 P_k}{\partial\theta_0(\partial\boldsymbol{\alpha}_{0(k)})^{<2>}}\frac{\partial^2 P_k}{\partial\theta_0^2}\right.\right.$$

$$\left.+\sum_{\otimes}^{2}\frac{\partial^3 P_k}{\partial\theta_0^2\partial\boldsymbol{\alpha}_{0(k)}}\otimes\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_{0(k)}}+\frac{\partial P_k}{\partial\theta_0}\frac{\partial^4 P_k}{\partial\theta_0^2(\partial\boldsymbol{\alpha}_{0(k)})^{<2>}}\right)$$

$$+\frac{1}{(P_kQ_k)^2}\left\{-(1-2P_k)\left\{2\sum_{\otimes}^{2}\left(\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_{0(k)}}\frac{\partial^2 P_k}{\partial\theta_0^2}+\frac{\partial^3 P_k}{\partial\theta_0^2\partial\boldsymbol{\alpha}_{0(k)}}\frac{\partial P_k}{\partial\theta_0}\right)\right.\right.$$

$$\otimes\frac{\partial P_k}{\partial\boldsymbol{\alpha}_{0(k)}}+2\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{\partial\theta_0^2}\frac{\partial^2 P_k}{(\partial\boldsymbol{\alpha}_{0(k)})^{<2>}}$$

$$\left.+6\left(\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_{0(k)}}\right)^{<2>}\frac{\partial P_k}{\partial\theta_0}+3\left(\frac{\partial P_k}{\partial\theta_0}\right)^2\frac{\partial^3 P_k}{\partial\theta_0(\partial\boldsymbol{\alpha}_{0(k)})^{<2>}}\right\}$$

$$+4\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{\partial\theta_0^2}\left(\frac{\partial P_k}{\partial\boldsymbol{\alpha}_{0(k)}}\right)^{<2>}$$

$$\left.+6\sum_{\otimes}^{2}\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_{0(k)}}\otimes\frac{\partial P_k}{\partial\boldsymbol{\alpha}_{0(k)}}\left(\frac{\partial P_k}{\partial\theta_0}\right)^2+2\left(\frac{\partial P_k}{\partial\theta_0}\right)^3\frac{\partial^2 P_k}{(\partial\boldsymbol{\alpha}_{0(k)})^{<2>}}\right\}$$

$$+\frac{1}{(P_kQ_k)^3}\left\{2(1-2P_k)^2\left\{2\frac{\partial P_k}{\partial\theta_0}\frac{\partial^2 P_k}{\partial\theta_0^2}\left(\frac{\partial P_k}{\partial\boldsymbol{\alpha}_{0(k)}}\right)^{<2>}\right.\right.$$

$$\left.+3\left(\frac{\partial P_k}{\partial\theta_0}\right)^2\sum_{\otimes}^{2}\frac{\partial^2 P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_{0(k)}}\otimes\frac{\partial P_k}{\partial\boldsymbol{\alpha}_{0(k)}}+2\left(\frac{\partial P_k}{\partial\theta_0}\right)^3\frac{\partial^2 P_k}{(\partial\boldsymbol{\alpha}_{0(k)})^{<2>}}\right\}$$

$$\left.\left.-12(1-2P_k)\left(\frac{\partial P_k}{\partial\theta_0}\right)^3\left(\frac{\partial P_k}{\partial\boldsymbol{\alpha}_{0(k)}}\right)^{<2>}\right\}-\frac{6(1-2P_k)^3}{(P_kQ_k)^4}\left(\frac{\partial P_k}{\partial\theta_0}\right)^3\left(\frac{\partial P_k}{\partial\boldsymbol{\alpha}_{0(k)}}\right)^{<2>}\right].$$

(a.2) $$i_{\theta_0}^{(2)}=\frac{\overline{i}_{\theta_0}^{-1/2}}{4}\overline{i}_{\theta_0}^{(\mathrm{D2})}-\frac{\overline{i}_{\theta_0}^{-3/2}}{8}(\overline{i}_{\theta_0}^{(\mathrm{D1})})^2$$

$$\frac{\partial i_{\theta_0}^{(2)}}{\partial \theta_0} = -\frac{3}{8}\overline{i}_{\theta_0}^{-3/2}\overline{i}_{\theta_0}^{(\mathrm{D1})}\overline{i}_{\theta_0}^{(\mathrm{D2})} + \frac{\overline{i}_{\theta_0}^{-1/2}}{4}\overline{i}_{\theta_0}^{(\mathrm{D3})} + \frac{3}{16}\overline{i}_{\theta_0}^{-5/2}(\overline{i}_{\theta_0}^{(\mathrm{D1})})^3,$$

$$\frac{\partial i_{\theta_0}^{(2)}}{\partial \boldsymbol{\alpha}_0} = -\frac{\overline{i}_{\theta_0}^{-3/2}}{8}\frac{\partial \overline{i}_{\theta_0}}{\partial \boldsymbol{\alpha}_0}\overline{i}_{\theta_0}^{(\mathrm{D2})} + \frac{\overline{i}_{\theta_0}^{-1/2}}{4}\frac{\partial \overline{i}_{\theta_0}^{(\mathrm{D2})}}{\partial \boldsymbol{\alpha}_0} + \frac{3}{16}\overline{i}_{\theta_0}^{-5/2}\frac{\partial \overline{i}_{\theta_0}}{\partial \boldsymbol{\alpha}_0}(\overline{i}_{\theta_0}^{(\mathrm{D1})})^2$$

$$-\frac{\overline{i}_{\theta_0}^{-3/2}}{4}\overline{i}_{\theta_0}^{(\mathrm{D1})}\frac{\partial \overline{i}_{\theta_0}^{(\mathrm{D1})}}{\partial \boldsymbol{\alpha}_0},$$

where

$$\frac{\partial \overline{i}_{\theta_0}^{(\mathrm{D2})}}{\partial \boldsymbol{\alpha}_{0(k)}} = n^{-1}\frac{\partial}{\partial \boldsymbol{\alpha}_{0(k)}}\left[\frac{2}{P_kQ_k}\left\{\frac{\partial P_k}{\partial \theta_0}\frac{\partial^3 P_k}{\partial \theta_0^3} + \left(\frac{\partial^2 P_k}{\partial \theta_0^2}\right)^2\right\}\right.$$

$$+\frac{1}{(P_kQ_k)^2}\left\{-5(1-2P_k)\left(\frac{\partial P_k}{\partial \theta_0}\right)^2\frac{\partial^2 P_k}{\partial \theta_0^2} + 2\left(\frac{\partial P_k}{\partial \theta_0}\right)^4\right\}$$

$$\left.+\frac{1}{(P_kQ_k)^3}2(1-2P_k)^2\left(\frac{\partial P_k}{\partial \theta_0}\right)^4\right]$$

$$= n^{-1}\left[\frac{2}{P_kQ_k}\left(\frac{\partial^2 P_k}{\partial \theta_0 \partial \boldsymbol{\alpha}_{0(k)}}\frac{\partial^3 P_k}{\partial \theta_0^3} + \frac{\partial P_k}{\partial \theta_0}\frac{\partial^4 P_k}{\partial \theta_0^3 \partial \boldsymbol{\alpha}_{0(k)}} + 2\frac{\partial^2 P_k}{\partial \theta_0^2}\frac{\partial^3 P_k}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_{0(k)}}\right)\right.$$

$$+\frac{1}{(P_kQ_k)^2}\left\{(1-2P_k)\left\{-2\left(\frac{\partial P_k}{\partial \theta_0}\frac{\partial^3 P_k}{\partial \theta_0^3} + \left(\frac{\partial P_k}{\partial \theta_0}\right)^2\right)\frac{\partial P_k}{\partial \boldsymbol{\alpha}_{0(k)}}\right.\right.$$

$$\left.\left.-5\left(2\frac{\partial P_k}{\partial \theta_0}\frac{\partial^2 P_k}{\partial \theta_0^2}\frac{\partial^2 P_k}{\partial \theta_0 \partial \boldsymbol{\alpha}_{0(k)}} + \left(\frac{\partial P_k}{\partial \theta_0}\right)^2\frac{\partial^3 P_k}{\partial \theta_0^2 \partial \boldsymbol{\alpha}_{0(k)}}\right)\right\}\right.$$

$$+10\left(\frac{\partial P_k}{\partial \theta_0}\right)^2 \frac{\partial^2 P_k}{\partial \theta_0^2}\frac{\partial P_k}{\partial \boldsymbol{\alpha}_{0(k)}} + 8\left(\frac{\partial P_k}{\partial \theta_0}\right)^3 \frac{\partial^2 P_k}{\partial \theta_0 \partial \boldsymbol{\alpha}_{0(k)}}\Bigg\}$$

$$+\frac{1}{(P_k Q_k)^3}\Bigg\{(1-2P_k)^2\Bigg\{10\left(\frac{\partial P_k}{\partial \theta_0}\right)^2 \frac{\partial^2 P_k}{\partial \theta_0^2}\frac{\partial P_k}{\partial \boldsymbol{\alpha}_{0(k)}} + 8\left(\frac{\partial P_k}{\partial \theta_0}\right)^3 \frac{\partial^2 P_k}{\partial \theta_0 \partial \boldsymbol{\alpha}_{0(k)}}\Bigg\}$$

$$-12(1-2P_k)\left(\frac{\partial P_k}{\partial \theta_0}\right)^4 \frac{\partial P_k}{\partial \boldsymbol{\alpha}_{0(k)}}\Bigg\} - \frac{6(1-2P_k)^3}{(P_k Q_k)^4}\left(\frac{\partial P_k}{\partial \theta_0}\right)^4 \frac{\partial P_k}{\partial \boldsymbol{\alpha}_{0(k)}}\Bigg].$$

(a.3) $$\bar{\beta}_{2I} = \bar{i}_{\theta_0}^{-1} + c\bar{i}_{\theta_0}^{-2}\mathrm{E}_{\theta_0}\left(-\frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0'}\right)\{\mathrm{E}_{\boldsymbol{\alpha}_0}(\mathbf{G}_0)\}^{-1}\mathrm{E}_{\theta_0}\left(-\frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0}\right)$$

Note that $\bar{i}_{\theta_0}$ and $\mathrm{E}_{\theta_0}(\cdot)$ in $\bar{\beta}_{2I}$ include θ_0 and $\boldsymbol{\alpha}_0$ while $\mathrm{E}_{\boldsymbol{\alpha}_0}(\mathbf{G}_0) \equiv \boldsymbol{\Gamma}_{\mathbf{G}_0} \equiv \mathbf{I}_{\boldsymbol{\alpha}_0}$ includes only $\boldsymbol{\alpha}_0$. Define $\mathrm{E}_{\theta_0}\left(-\frac{\partial^2 \bar{l}_{\theta_0}}{\partial \theta_0 \partial \boldsymbol{\alpha}_0}\right) = n^{-1}\sum_{k=1}^{n}\frac{1}{P_k Q_k}\frac{\partial P_k}{\partial \theta_0}\frac{\partial P_k}{\partial \boldsymbol{\alpha}_0} \equiv \mathbf{d}$. Let $\alpha_{0(k)}$ and $\alpha_{0(k*)}$ be possibly different elements in $\boldsymbol{\alpha}_{0(k)}$ (recall that $\boldsymbol{\alpha}_0 = (\boldsymbol{\alpha}_{0(1)}', \ldots, \boldsymbol{\alpha}_{0(n)}')'$); and let $\gamma_{jk}^{(\mathbf{G}_0)} = (\boldsymbol{\Gamma}_{\mathbf{G}_0})_{jk}$ $(q \geq j \geq k \geq 1)$ or the (*j*, *k*)th element of $\boldsymbol{\Gamma}_{\mathbf{G}_0}$ with *q* being the number of item parameters. Then,

$$\bar{\beta}_{2I} = \bar{i}_{\theta_0}^{-1} + c\bar{i}_{\theta_0}^{-2}\mathbf{d}'\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d},$$

$$\frac{\partial \bar{\beta}_{2I}}{\partial \theta_0} = -(\bar{i}_{\theta_0}^{-2} + 2c\bar{i}_{\theta_0}^{-3}\mathbf{d}'\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})\bar{i}_{\theta_0}^{(\mathrm{D1})} + 2c\bar{i}_{\theta_0}^{-2}\mathbf{d}'\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial \mathbf{d}}{\partial \theta_0},$$

$$\frac{\partial \bar{\beta}_{2I}}{\partial \alpha_{0(k)}} = -(\bar{i}_{\theta_0}^{-2} + 2c\bar{i}_{\theta_0}^{-3}\mathbf{d}'\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})\frac{\partial \bar{i}_{\theta_0}}{\partial \alpha_{0(k)}} + 2c\bar{i}_{\theta_0}^{-2}\mathbf{d}'\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial \mathbf{d}}{\partial \alpha_{0(k)}}$$

$$-c\bar{i}_{\theta_0}^{-2}\mathbf{d}'\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial \boldsymbol{\Gamma}_{\mathbf{G}_0}}{\partial \alpha_{0(k)}}\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d} \quad (k = 1, .., n),$$

$$\frac{\partial \overline{\beta}_{2I}}{\partial \gamma_{jk}^{(\mathbf{G}_0)}} = -\overline{c}\overline{i}_{\theta_0}^{-2}\mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\frac{2-\delta_{jk}}{2}(\mathbf{E}_{jk}+\mathbf{E}_{jk})\ \mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d}$$

$$= -\overline{c}\overline{i}_{\theta_0}^{-2}(2-\delta_{jk})(\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})_j(\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})_k \quad (q \geq j \geq k \geq 1),$$

where δ_{jk} is the Kronecker delta, $\mathbf{E}_{jk}$ is the matrix of an appropriate size whose (*j*, *k*)the element is 1 with the remaining ones being 0, and $(\cdot)_j$ denote the *j*-th element of a vector.

The matrix $\frac{\partial \mathbf{G}_0}{\partial \alpha_{0(k)}} = N^{-1}\sum_{j=1}^{N}\left(\frac{\partial^2 l_{\boldsymbol{\alpha}(j)}}{\partial \boldsymbol{\alpha}_0 \partial \alpha_{0(k)}}\frac{\partial l_{\boldsymbol{\alpha}(j)}}{\partial \boldsymbol{\alpha}_0'} + \frac{\partial l_{\boldsymbol{\alpha}(j)}}{\partial \boldsymbol{\alpha}_0}\frac{\partial^2 l_{\boldsymbol{\alpha}(j)}}{\partial \boldsymbol{\alpha}_0' \partial \alpha_{0(k)}}\right)$ is also used when evaluating the sampling behavior of stochastic $\mathbf{G}_0 = O_p(1)$ including $\boldsymbol{\alpha}_0$ and $\hat{\mathbf{G}} = O_p(1)$ using $\hat{\boldsymbol{\alpha}}$ (recall (4.3); see also Ogasawara, 2010).

$$\frac{\partial \mathbf{d}}{\partial \theta_0} = n^{-1}\sum_{k=1}^{n}\left\{\frac{1}{P_kQ_k}\left(\frac{\partial^2 P_k}{\partial \theta_0^2}\frac{\partial P_k}{\partial \boldsymbol{\alpha}_0} + \frac{\partial P_k}{\partial \theta_0}\frac{\partial^2 P_k}{\partial \theta_0 \partial \boldsymbol{\alpha}_0}\right) - \frac{1-2P_k}{(P_kQ_k)^2}\left(\frac{\partial P_k}{\partial \theta_0}\right)^2\frac{\partial P_k}{\partial \boldsymbol{\alpha}_0}\right\},$$

$$\frac{\partial \mathbf{d}}{\partial \alpha_{0(k)}} = n^{-1}\left\{\frac{1}{P_kQ_k}\left(\frac{\partial^2 P_k}{\partial \theta_0 \partial \alpha_{0(k)}}\frac{\partial P_k}{\partial \boldsymbol{\alpha}_0} + \frac{\partial P_k}{\partial \theta_0}\frac{\partial^2 P_k}{\partial \boldsymbol{\alpha}_0 \partial \alpha_{0(k)}}\right)\right.$$

$$\left. - \frac{1-2P_k}{(P_kQ_k)^2}\frac{\partial P_k}{\partial \theta_0}\frac{\partial P_k}{\partial \boldsymbol{\alpha}_0}\frac{\partial P_k}{\partial \alpha_{0(k)}}\right\} \quad (k = 1,\ldots,n),$$

$$\frac{\partial^2 \overline{\beta}_{2I}}{\partial \theta_0^2} = (2\overline{i}_{\theta_0}^{-3} + 6\overline{c}\overline{i}_{\theta_0}^{-4}\mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})(\overline{i}_{\theta_0}^{(\mathrm{D1})})^2 - 8\overline{c}\overline{i}_{\theta_0}^{-3}\mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial \mathbf{d}}{\partial \theta_0}\overline{i}_{\theta_0}^{(\mathrm{D1})}$$

$$-(\overline{i}_{\theta_0}^{-2} + 2\overline{c}\overline{i}_{\theta_0}^{-3}\mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})\overline{i}_{\theta_0}^{(\mathrm{D2})} + 2\overline{c}\overline{i}_{\theta_0}^{-2}\left(\frac{\partial \mathbf{d}'}{\partial \theta_0}\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial \mathbf{d}}{\partial \theta_0} + \mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial^2 \mathbf{d}}{\partial \theta_0^2}\right),$$

$$\frac{\partial^2 \overline{\beta}_{2I}}{\partial\theta_0 \partial\alpha_{0(k)}} = (2\overline{i}_{\theta_0}^{-3} + 6\overline{c}\overline{i}_{\theta_0}^{-4}\mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})\overline{i}_{\theta_0}^{(\mathrm{D1})}\frac{\partial \overline{i}_{\theta_0}}{\partial\alpha_{0(k)}}$$

$$-4\overline{c}\overline{i}_{\theta_0}^{-3}\left(\mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\mathbf{d}}{\partial\alpha_{0(k)}}\overline{i}_{\theta_0}^{(\mathrm{D1})} + \mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\mathbf{d}}{\partial\theta_0}\frac{\partial \overline{i}_{\theta_0}}{\partial\alpha_{0(k)}}\right)$$

$$-(\overline{i}_{\theta_0}^{-2} + 2\overline{c}\overline{i}_{\theta_0}^{-3}\mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})\frac{\partial \overline{i}_{\theta_0}^{(\mathrm{D1})}}{\partial\alpha_{0(k)}} + 2\overline{c}\overline{i}_{\theta_0}^{-3}\mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\mathbf{\Gamma}_{\mathbf{G}_0}}{\partial\alpha_{0(k)}}\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d}\overline{i}_{\theta_0}^{(\mathrm{D1})}$$

$$+2\overline{c}\overline{i}_{\theta_0}^{-2}\left(\frac{\partial\mathbf{d}'}{\partial\alpha_{0(k)}}\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\mathbf{d}}{\partial\theta_0} + \mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial^2\mathbf{d}}{\partial\theta_0\partial\alpha_{0(k)}} - \mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\mathbf{\Gamma}_{\mathbf{G}_0}}{\partial\alpha_{0(k)}}\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\mathbf{d}}{\partial\theta_0}\right)$$

$$(k = 1,\ldots,n),$$

$$\frac{\partial^2 \overline{\beta}_{2I}}{\partial\theta_0 \partial\gamma_{jk}^{(\mathbf{G}_0)}} = 2\overline{c}\overline{i}_{\theta_0}^{-3}(2-\delta_{jk})(\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})_j(\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})_k\,\overline{i}_{\theta_0}^{(\mathrm{D1})}$$

$$-\overline{c}\overline{i}_{\theta_0}^{-2}(2-\delta_{jk})\left\{\left(\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\mathbf{d}}{\partial\theta_0}\right)_j\left(\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d}\right)_k + \left(\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d}\right)_j\left(\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\mathbf{d}}{\partial\theta_0}\right)_k\right\}$$

$$(q \geq j \geq k \geq 1),$$

$$\frac{\partial^2 \overline{\beta}_{2I}}{\partial\alpha_{0(j)}\partial\alpha_{0(k)}} = -(\overline{i}_{\theta_0}^{-2} + 2\overline{c}\overline{i}_{\theta_0}^{-3}\mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})\frac{\partial^2 \overline{i}_{\theta_0}}{\partial\alpha_{0(j)}\partial\alpha_{0(k)}}$$

$$+(2\overline{i}_{\theta_0}^{-3} + 6\overline{c}\overline{i}_{\theta_0}^{-4}\mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})\frac{\partial \overline{i}_{\theta_0}}{\partial\alpha_{0(j)}}\frac{\partial \overline{i}_{\theta_0}}{\partial\alpha_{0(k)}}$$

$$-4\overline{c}\overline{i}_{\theta_0}^{-3}\left(\mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\mathbf{d}}{\partial\alpha_{0(j)}}\frac{\partial \overline{i}_{\theta_0}}{\partial\alpha_{0(k)}} + \mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\mathbf{d}}{\partial\alpha_{0(k)}}\frac{\partial \overline{i}_{\theta_0}}{\partial\alpha_{0(j)}}\right)$$

$$+\sum_{(j,k)}^{2} 2\overline{c}\overline{i}_{\theta_0}^{-3}\mathbf{d}'\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\mathbf{\Gamma}_{\mathbf{G}_0}}{\partial\alpha_{0(j)}}\mathbf{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d}\frac{\partial \overline{i}_{\theta_0}}{\partial\alpha_{0(k)}}$$

$$+\overline{c}\,\overline{i}_{\theta_0}^{-2}\left(-\sum_{(j,k)}^{2} 2\mathbf{d}'\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\boldsymbol{\Gamma}_{\mathbf{G}_0}}{\partial\alpha_{0(j)}}\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\mathbf{d}}{\partial\alpha_{0(k)}}+2\frac{\partial^2\mathbf{d}}{\partial\alpha_{0(j)}\partial\alpha_{0(k)}}\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d}\right.$$

$$+2\frac{\partial\mathbf{d}'}{\partial\alpha_{0(j)}}\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\mathbf{d}}{\partial\alpha_{0(k)}}+2\mathbf{d}'\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\boldsymbol{\Gamma}_{\mathbf{G}_0}}{\partial\alpha_{0(j)}}\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\boldsymbol{\Gamma}_{\mathbf{G}_0}}{\partial\alpha_{0(k)}}\mathbf{d}$$

$$\left.-\mathbf{d}'\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial^2\boldsymbol{\Gamma}_{\mathbf{G}_0}}{\partial\alpha_{0(j)}\partial\alpha_{0(k)}}\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d}\right)\quad (j,k=1,\ldots,n),$$

$$\frac{\partial^2\overline{\beta}_{2I}}{\partial\gamma_{jk}^{(\mathbf{G}_0)}\partial\alpha_{0(l^*)}}=2\overline{c}\,\overline{i}_{\theta_0}^{-3}(2-\delta_{jk})(\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})_j(\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})_k\frac{\partial\overline{i}_{\theta_0}}{\partial\alpha_{0(l^*)}}$$

$$+\overline{c}\,\overline{i}_{\theta_0}^{-2}(2-\delta_{jk})\sum_{(j,k)}^{2}\left(\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\boldsymbol{\Gamma}_{\mathbf{G}_0}}{\partial\alpha_{0(l^*)}}\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d}-\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\frac{\partial\mathbf{d}}{\partial\alpha_{0(l^*)}}\right)_j(\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})_k$$

$$(q\ge j\ge k\ge 1; l^*=1,\ldots,n),$$

$$\frac{\partial^2\overline{\beta}_{2I}}{\partial\gamma_{jk}^{(\mathbf{G}_0)}\partial\gamma_{l^*m^*}^{(\mathbf{G}_0)}}=\frac{\overline{c}}{2}\overline{i}_{\theta_0}^{-2}(2-\delta_{jk})(2-\delta_{l^*m^*})$$

$$\times\sum_{(j,k)}^{2}\sum_{(l^*,m^*)}^{2}(\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1})_{jl^*}(\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})_{m^*}(\boldsymbol{\Gamma}_{\mathbf{G}_0}^{-1}\mathbf{d})_k$$

$$(q\ge j\ge k\ge 1;\ q\ge l^*\ge m^*\ge 1),$$

where

$$\frac{\partial^2\mathbf{d}}{\partial\theta_0^2}=n^{-1}\sum_{k=1}^{n}\left[\frac{1}{P_kQ_k}\left(\frac{\partial^3P_k}{\partial\theta_0^3}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}+2\frac{\partial^2P_k}{\partial\theta_0^2}\frac{\partial^2P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}+\frac{\partial P_k}{\partial\theta_0}\frac{\partial^3P_k}{\partial\theta_0^2\partial\boldsymbol{\alpha}_0}\right)\right.$$

$$+\frac{1}{(P_kQ_k)^2}\left\{-(1-2P_k)\left(3\frac{\partial^2P_k}{\partial\theta_0^2}\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}+2\left(\frac{\partial P_k}{\partial\theta_0}\right)^2\frac{\partial^2P_k}{\partial\theta_0\partial\boldsymbol{\alpha}_0}\right)\right.$$

$$\left.\left.+2\left(\frac{\partial P_k}{\partial\theta_0}\right)^3\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\right\}+\frac{2(1-2P_k)^2}{(P_kQ_k)^3}\left(\frac{\partial P_k}{\partial\theta_0}\right)^3\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\right],$$

$$\frac{\partial^2 \mathbf{d}}{\partial\theta_0 \partial\alpha_{0(k)}} = n^{-1}\left[\frac{1}{P_k Q_k}\left(\frac{\partial^3 P_k}{\partial\theta_0^2 \partial\alpha_{0(k)}} \frac{\partial P_k}{\partial \boldsymbol{\alpha}_0} + \frac{\partial^2 P_k}{\partial\theta_0^2} \frac{\partial^2 P_k}{\partial\boldsymbol{\alpha}_0 \partial\alpha_{0(k)}} \right.\right.$$

$$\left. + \frac{\partial^2 P_k}{\partial\theta_0 \partial\alpha_{0(k)}} \frac{\partial^2 P_k}{\partial\theta_0 \partial\boldsymbol{\alpha}_0} + \frac{\partial P_k}{\partial\theta_0} \frac{\partial^3 P_k}{\partial\theta_0 \partial\boldsymbol{\alpha}_0 \partial\alpha_{0(k)}} \right)$$

$$+ \frac{1}{(P_k Q_k)^2}\left\{ -(1-2P_k)\left\{ \left(\frac{\partial^2 P_k}{\partial\theta_0^2} \frac{\partial P_k}{\partial\boldsymbol{\alpha}_0} + \frac{\partial P_k}{\partial\theta_0} \frac{\partial^3 P_k}{\partial\theta_0 \partial\boldsymbol{\alpha}_0} \right) \frac{\partial P_k}{\partial\alpha_{0(k)}} \right.\right.$$

$$\left. + 2\frac{\partial P_k}{\partial\theta_0} \frac{\partial^2 P_k}{\partial\theta_0 \partial\alpha_{0(k)}} \frac{\partial P_k}{\partial\boldsymbol{\alpha}_0} + \left(\frac{\partial P_k}{\partial\theta_0} \right)^2 \frac{\partial^2 P_k}{\partial\boldsymbol{\alpha}_0 \partial\alpha_{0(k)}} \right\}$$

$$\left.\left. + 2\left(\frac{\partial P_k}{\partial\theta_0} \right)^2 \frac{\partial P_k}{\partial\boldsymbol{\alpha}_0} \frac{\partial P_k}{\partial\alpha_{0(k)}} \right\} + \frac{2(1-2P_k)^2}{(P_k Q_k)^3}\left(\frac{\partial P_k}{\partial\theta_0} \right)^2 \frac{\partial P_k}{\partial\boldsymbol{\alpha}_0} \frac{\partial P_k}{\partial\alpha_{0(k)}} \right]$$

$$(k = 1, \ldots, n),$$

$$\frac{\partial^2 \mathbf{d}}{\partial\alpha_{0(k)} \partial\alpha_{0(k^*)}} = n^{-1}\left[\frac{1}{P_k Q_k}\left(\frac{\partial^3 P_k}{\partial\theta_0 \partial\alpha_{0(k)} \partial\alpha_{0(k^*)}} \frac{\partial P_k}{\partial\boldsymbol{\alpha}_0} \right.\right.$$

$$\left. + \sum_{(k,k^*)}^{2} \frac{\partial^2 P_k}{\partial\theta_0 \partial\alpha_{0(k)}} \frac{\partial^2 P_k}{\partial\boldsymbol{\alpha}_0 \partial\alpha_{0(k^*)}} + \frac{\partial P_k}{\partial\theta_0} \frac{\partial^3 P_k}{\partial\boldsymbol{\alpha}_0 \partial\alpha_{0(k)} \partial\alpha_{0(k^*)}} \right)$$

$$+ \frac{1}{(P_k Q_k)^2}\left\{ -(1-2P_k)\left\{ \sum_{(k,k^*)}^{2} \left(\frac{\partial P_k}{\partial\alpha_{0(k)}} \frac{\partial^2 P_k}{\partial\theta_0 \partial\alpha_{0(k^*)}} \frac{\partial P_k}{\partial\boldsymbol{\alpha}_0} \right.\right.\right.$$

$$\left. + \frac{\partial P_k}{\partial\theta_0} \frac{\partial P_k}{\partial\alpha_{0(k)}} \frac{\partial^2 P_k}{\partial\boldsymbol{\alpha}_0 \partial\alpha_{0(k^*)}} \right)$$

$$+\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\frac{\partial^2 P_k}{\partial\alpha_{0(k)}\partial\alpha_{0(k^*)}}\Bigg\}+2\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\frac{\partial P_k}{\partial\alpha_{0(k)}}\frac{\partial P_k}{\partial\alpha_{0(k^*)}}\Bigg\}$$

$$+\frac{2(1-2P_k)^3}{(P_kQ_k)^3}\frac{\partial P_k}{\partial\theta_0}\frac{\partial P_k}{\partial\boldsymbol{\alpha}_0}\frac{\partial P_k}{\partial\alpha_{0(k)}}\frac{\partial P_k}{\partial\alpha_{0(k^*)}}\Bigg]\ (k=1,...,n),$$

$$\frac{\partial^2\mathbf{G}_0}{\partial\alpha_{0(j)}\partial\alpha_{0(k)}}=N^{-1}\sum_{m^*=1}^{N}\Bigg(\frac{\partial^3 l_{\boldsymbol{\alpha}(m^*)}}{\partial\boldsymbol{\alpha}_0\partial\alpha_{0(j)}\partial\alpha_{0(k)}}\frac{\partial l_{\boldsymbol{\alpha}(m^*)}}{\partial\boldsymbol{\alpha}_0{}'}$$

$$+\sum_{(j,k)}^{2}\frac{\partial^2 l_{\boldsymbol{\alpha}(m^*)}}{\partial\boldsymbol{\alpha}_0\partial\alpha_{0(j)}}\frac{\partial^2 l_{\boldsymbol{\alpha}(m^*)}}{\partial\boldsymbol{\alpha}_0{}'\partial\alpha_{0(k)}}+\frac{\partial l_{\boldsymbol{\alpha}(m^*)}}{\partial\boldsymbol{\alpha}_0}\frac{\partial^3 l_{\boldsymbol{\alpha}(m^*)}}{\partial\boldsymbol{\alpha}_0{}'\partial\alpha_{0(j)}\partial\alpha_{0(k)}}\Bigg)$$

$(j,k=1,..,n).$

When the 3PLM is used, recall that $P_k=c_k+\frac{1-c_k}{1+\exp\{-Da_k(\theta_0-b_k)\}}=c_k+(1-c_k)B_k\ (k=1,...,n)$ and $D=1.7$. Then,

$$\frac{\partial P_k}{\partial\theta_0}=(1-c_k)Da_kB_k(1-B_k),$$

$$\frac{\partial^2 P_k}{\partial\theta_0^2}=(1-c_k)(Da_k)^2(1-2B_k)B_k(1-B_k),$$

$$\frac{\partial^3 P_k}{\partial\theta_0^3}=(1-c_k)(Da_k)^3(1-6B_k+6B_k^2)B_k(1-B_k).$$

Reference

Ogasawara, H. (2013). Asymptotic cumulants of ability estimators using fallible item parameters. *Journal of Multivariate Analysis, 119,* 144-162.

Table A1. Simulated and asymptotic standard errors of the studentized $\hat{\theta}$ when the 2PLM holds, where the item parameters are known or estimated by MML (n = 50)

Standard error		$t^*_{\boldsymbol{\alpha}_0} = n^{1/2}(\hat{\theta}-\theta_0)\hat{i}^{1/2}_{\boldsymbol{\alpha}_0}$			$t = n^{1/2}(\hat{\theta}-\theta_0)\hat{\bar{\beta}}^{-1/2}_{2I}$			
					Estimated item parameters			
		Known item parameters			N = 500		N = 1,000	
		SD$^{(t0)}$	ASE$^{(t0)}$	HASE$^{(t0)}$	SD$^{(t1)}$	ASE$^{(t1)}$	SD$^{(t1)}$	ASE$^{(t1)}$
θ = -1	ML	.989	1	.999	1.044	1	.984	1
	BM	.897	1	.883	.946	1	.889	1
	WL	.987	1	1.000	1.043	1	.983	1
θ = 0	ML	1.006	1	.992	1.055	1	.996	1
	BM	.938	1	.916	.979	1	.924	1
	WL	.993	1	.978	1.043	1	.984	1
θ = 1	ML	.996	1	.992	1.016	1	.964	1
	BM	.932	1	.919	.949	1	.899	1
	WL	.980	1	.976	1.002	1	.950	1
θ = 2	ML	.976	1	.981	.995	1	1.014	1
	BM	.893	1	.866	.914	1	.929	1
	WL	.956	1	.953	.979	1	.997	1

Note. SD$^{(t0)}$ (SD$^{(t1)}$) = the standard deviation from simulations, ASE$^{(t0)}$ = $(\beta_2^{(0)}\bar{i}_{\theta_0})^{1/2}$, ASE$^{(t1)} = \bar{\beta}_{t2}^{1/2} = (\bar{\beta}_2\bar{\beta}_{2I}^{-1})^{1/2}$, HASE$^{(t0)}$ = $(\beta_2^{(0)}\bar{i}_{\theta_0} + n^{-1}\beta_{tH2}^{(0)})^{1/2} = \{(\text{ASE}^{(t0)})^2 + n^{-1}\beta_{tH2}^{(0)}\}^{1/2}$, HASE$^{(t1)} = (\bar{\beta}_{t2}\bar{i}_{\theta_0} + n^{-1}\beta_{tH2}^{(0)})^{1/2} = \{(\text{ASE}^{(t1)})^2 + n^{-1}\beta_{tH2}^{(0)}\}^{1/2}$. Under c.m.s., ASE$^{(t0)}$ = ASE$^{(t1)}$ = 1 and HASE$^{(t0)}$ = HASE$^{(t1)}$ = $(1 + n^{-1}\beta_{tH2}^{(0)})^{1/2}$. The $\hat{i}_{\boldsymbol{\alpha}_0}$ in $t^*_{\boldsymbol{\alpha}_0}$ is given by $\hat{\theta}$ and known $\boldsymbol{\alpha}_0$ ($t^*_{\boldsymbol{\alpha}_0}$ and $\hat{i}_{\boldsymbol{\alpha}_0}$ should not be confused with t^* and $\hat{i}$ using $\hat{\theta}$ and estimated $\hat{\boldsymbol{\alpha}}$ in the text pages). The numbers of deleted cases are the same as those of Table 2. See also the footnote of Table 1.

Table A2. Simulated and asymptotic biases of the studentized $\hat{\theta}$ when the 2PLM holds, where the item parameters are known or estimated by MML (n = 50)

Bias		$t^*_{\boldsymbol{\alpha}_0} = n^{1/2}(\hat{\theta}-\theta_0)\hat{i}^{1/2}_{\boldsymbol{\alpha}_0}$		$t = n^{1/2}(\hat{\theta}-\theta_0)\hat{\bar{\beta}}^{-1/2}_{2I}$			
		Known item parameters		Estimated item parameters			
				N = 500		N = 1,000	
		Sim.	Th.	Sim.	Th.	Sim.	Th.
θ = -1	ML	-.09	0	.10	.02	-.26	.01
	BM	2.05	2.33	2.23	2.36	1.92	2.34
	WL	.40	.47	.56	.50	.20	.49
θ = 0	ML	.04	0	-.12	.00	.04	.00
	BM	.06	0	-.10	.00	.05	.00
	WL	.32	.30	.15	.31	.31	.30
θ = 1	ML	.03	0	.68	-.02	1.18	-.01
	BM	-1.71	-1.86	-1.12	-1.88	-.68	-1.87
	WL	-.16	-.20	.49	-.22	.99	-.21
θ = 2	ML	.00	0	.72	-.06	1.74	-.03
	BM	-4.16	-4.61	-3.44	-4.66	-2.58	-4.63
	WL	-.73	-.76	.03	-.82	1.04	-.79

Note. Sim.= $n^{1/2}$ times the simulated bias, Th. = $\bar{\beta}_{t1}$ ($n^{1/2}$times the theoretical or asymptotic bias). See also the footnote of Table 1.

Table A3. Simulated and asymptotic third cumulants of the studentized $\hat{\theta}$ when the 2PLM holds, where the item parameters are known or estimated by MML (n = 50)

Third cumulant		$t^*_{\boldsymbol{\alpha}_0} = n^{1/2}(\hat{\theta}-\theta_0)\hat{i}^{1/2}_{\boldsymbol{\alpha}_0}$		$t = n^{1/2}(\hat{\theta}-\theta_0)\hat{\bar{\beta}}^{-1/2}_{2I}$			
		Known item parameters		Estimated item parameters			
				N = 500		N = 1,000	
		Sim.	Th.	Sim.	Th.	Sim.	Th.
θ = -1	ML	1.14	.95	1.74	1.18	.62	1.06
	BM	.92	*	1.36	*	.50	*
	WL	1.18	*	1.80	*	.67	*
θ = 0	ML	.23	.60	.28	.67	1.14	.64
	BM	.29	*	.31	*	1.00	*
	WL	.11	*	.14	*	1.00	*
θ = 1	ML	-.29	-.41	.10	-.58	-.69	-.49
	BM	-.20	*	.16	*	-.52	*
	WL	-.24	*	.13	*	-.63	*
θ = 2	ML	-1.49	-1.52	-1.78	-2.17	-3.20	-1.85
	BM	-1.04	*	-1.19	*	-2.28	*
	WL	-1.32	*	-1.57	*	-2.93	*

Note. Sim.= $n^{1/2}$ times the simulated third cumulant, Th. = $\bar{\beta}_{t3}$ ($n^{1/2}$times the theoretical or asymptotic third cumulant). See also the footnote of Table 1.

Table A4. Simulated and asymptotic fourth cumulants of the studentized $\hat{\theta}$ when the 2PLM holds, where the item parameters are known or estimated by MML (n = 50)

Fourth cumulant		$t^*_{\boldsymbol{\alpha}_0} = n^{1/2}(\hat{\theta}-\theta_0)\hat{i}^{1/2}_{\boldsymbol{\alpha}_0}$		$t = n^{1/2}(\hat{\theta}-\theta_0)\hat{\bar{\beta}}^{-1/2}_{2I}$	
		Known item parameters		Estimated item parameters	
				N = 500	N = 1,000
		Sim.	Th.	Sim.	Sim.
θ = -1	ML	-1.18	.49	-13.31	-6.73
	BM	.51	*	-7.76	-3.26
	WL	-1.94	*	-14.07	-7.54
θ = 0	ML	-3.33	-4.23	-12.84	4.33
	BM	-2.57	*	-9.51	3.49
	WL	-2.83	*	-11.75	4.22
θ = 1	ML	-4.23	-4.65	-.39	2.03
	BM	-3.02	*	.58	1.72
	WL	-4.06	*	-.42	1.71
θ = 2	ML	-8.31	-8.00	-3.04	10.10
	BM	-4.71	*	-.58	8.12
	WL	-6.67	*	-2.04	10.42

Note. Sim.= n times the simulated fourth cumulant, Th. = $\bar{\beta}_{t4}$ (n times the theoretical or asymptotic fourth cumulant).
See also the footnote of Table 1.