A supplementary note on the paper
"Asymptotic expansion and asymptotic robustness
of the normal-theory estimators in the random regression model"

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This note is to supplement Ogasawara (2007), and gives (1) the matrix expressions of the partial derivatives of the parameter estimators with respect to sample variances and covariances, and (2) the direct proof of Theorem 1 using cumulants.

1. Matrix expressions of the partial derivatives
1.1 The regression coefficients
(a) The first partial derivatives

Let $\mathbf{s}_X = \mathrm{v}(\mathbf{S}_{XX})$, then

$$\frac{\partial \hat{\boldsymbol{\beta}}}{\partial \mathbf{s}_X{}'} = \frac{\partial (\mathbf{s}_{XY}{}' \otimes \mathbf{I}_p)\mathrm{vec}\mathbf{S}_{XX}^{-1}}{\partial \mathbf{s}_X{}'}$$

$$= -(\mathbf{s}_{XY}{}' \otimes \mathbf{I}_p)(\mathbf{S}_{XX}^{-1} \otimes \mathbf{S}_{XX}^{-1})\frac{\partial \mathrm{vec}\mathbf{S}_{XX}}{\partial \mathbf{s}_X{}'} = -\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1}) \otimes \mathbf{S}_{XX}^{-1}\}\mathbf{D}_p, \tag{A1.1}$$

$$\frac{\partial \hat{\boldsymbol{\beta}}}{\partial \mathbf{s}_{XY}{}'} = \mathbf{S}_{XX}^{-1},$$

where for $\mathrm{v}(\cdot)$, $\mathrm{vec}(\cdot)$, $\otimes$ and $\mathbf{D}_p$ see Section 4 of Ogasawara (2007).

(b) The nonzero second partial derivatives

Let $\mathbf{A}$ and $\mathbf{C}$ be $u \times v$ and $p \times q$ matrices, respectively. Then, the commutation matrix $\mathbf{K}_{uv}$ is defined as $\mathrm{vec}(\mathbf{A}') = \mathbf{K}_{uv}\mathrm{vec}(\mathbf{A})$ with $\mathbf{K}_u \equiv \mathbf{K}_{uu}$, which yields

$$\mathrm{vec}(\mathbf{A} \otimes \mathbf{C}) = (\mathbf{I}_v \otimes \mathbf{K}_{qu} \otimes \mathbf{I}_p)(\mathrm{vec}\mathbf{A} \otimes \mathrm{vec}\mathbf{C}), \tag{A1.2}$$

(Magnus & Neudecker, 1999, p.47), where $\mathbf{I}_v$ is the $v \times v$ identity matrix. Let $\mathbf{A}^{<k>} = \mathbf{A} \otimes \cdots \otimes \mathbf{A}$ (k times) and $\mathrm{vec}\mathbf{A} \otimes \cdots = (\mathrm{vec}\mathbf{A}) \otimes \cdots$, then

$$\partial \mathrm{vec}\left(\frac{\partial \hat{\boldsymbol{\beta}}}{\partial \mathbf{s}_X\,'}\right)\bigg/\partial \mathbf{s}_X\,' = -(\mathbf{D}_p\,'\otimes \mathbf{s}_{XY}\,'\otimes \mathbf{I}_p)\frac{\partial \mathrm{vec}(\mathbf{S}_{XX}^{-1}\otimes \mathbf{S}_{XX}^{-1})}{\partial \mathbf{s}_X\,'}$$

$$= -(\mathbf{D}_p\,'\otimes \mathbf{s}_{XY}\,'\otimes \mathbf{I}_p)(\mathbf{I}_p\otimes \mathbf{K}_p\otimes \mathbf{I}_p)\frac{\partial}{\partial \mathbf{s}_X\,'}(\mathrm{vec}\mathbf{S}_{XX}^{-1}\otimes \mathrm{vec}\mathbf{S}_{XX}^{-1})$$

$$= (\mathbf{D}_p\,'\otimes \mathbf{s}_{XY}\,'\otimes \mathbf{I}_p)(\mathbf{I}_p\otimes \mathbf{K}_p\otimes \mathbf{I}_p)$$

$$\quad \times\{(\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p)\otimes \mathrm{vec}\mathbf{S}_{XX}^{-1} + \mathrm{vec}\mathbf{S}_{XX}^{-1}\otimes(\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p)\},$$

(A1.3)

$$\partial \mathrm{vec}\left(\frac{\partial \hat{\boldsymbol{\beta}}}{\partial \mathbf{s}_X\,'}\right)\bigg/\partial \mathbf{s}_{XY}\,' = -(\mathbf{D}_p\,'\otimes \mathbf{I}_p)\frac{\partial \mathrm{vec}\{(\mathbf{s}_{XY}\,'\mathbf{S}_{XX}^{-1})\otimes \mathbf{S}_{XX}^{-1}\}}{\partial \mathbf{s}_{XY}\,'}$$

$$= -(\mathbf{D}_p\,'\otimes \mathbf{I}_p)\left\{\frac{\partial \mathrm{vec}(\mathbf{s}_{XY}\,'\mathbf{S}_{XX}^{-1})}{\partial \mathbf{s}_{XY}\,'}\otimes \mathrm{vec}\mathbf{S}_{XX}^{-1}\right\}$$

$$= -(\mathbf{D}_p\,'\otimes \mathbf{I}_p)(\mathbf{S}_{XX}^{-1}\otimes \mathrm{vec}\mathbf{S}_{XX}^{-1}).$$

(c) The nonzero third partial derivatives

Let $p^* = p(p+1)/2$, then

$$\partial \mathrm{vec}\left\{\partial \mathrm{vec}\left(\frac{\partial \hat{\boldsymbol{\beta}}}{\partial \mathbf{s}_X\,'}\right)\bigg/\partial \mathbf{s}_X\,'\right\}\bigg/\partial \mathbf{s}_X\,' = \partial \mathrm{vec}\left\{\frac{\partial^2 \hat{\boldsymbol{\beta}}}{(\partial \mathbf{s}_X\,')^{<2>}}\right\}\bigg/\partial \mathbf{s}_X\,'$$

$$= [\mathbf{I}_{p^*}\otimes\{(\mathbf{D}_p\,'\otimes \mathbf{s}_{XY}\,')(\mathbf{I}_p\otimes \mathbf{K}_p)\}\otimes \mathbf{I}_p]$$

$$\quad \times \partial \mathrm{vec}\{(\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p)\otimes \mathrm{vec}\mathbf{S}_{XX}^{-1} + \mathrm{vec}\mathbf{S}_{XX}^{-1}\otimes(\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p)\}/\partial \mathbf{s}_X\,',$$

(A1.4)

where

$$\partial \mathrm{vec}\{(\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p)\otimes \mathrm{vec}\mathbf{S}_{XX}^{-1} + \mathrm{vec}\mathbf{S}_{XX}^{-1}\otimes(\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p)\}/\partial \mathbf{s}_X\,'$$

$$= \partial[\mathrm{vec}(\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p)\otimes \mathrm{vec}\mathbf{S}_{XX}^{-1}$$

$$\quad + (\mathbf{K}_{p^*,p^2}\otimes \mathbf{I}_{p^2})\{\mathrm{vec}\mathbf{S}_{XX}^{-1}\otimes \mathrm{vec}(\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p)\}]/\partial \mathbf{s}_X\,'$$

$$= \partial[\{(\mathbf{D}_p\,'\otimes \mathbf{I}_{p^2})\mathrm{vec}\mathbf{S}_{XX}^{-1<2>}\}\otimes \mathrm{vec}\mathbf{S}_{XX}^{-1} + (\mathbf{K}_{p^*,p^2}\otimes \mathbf{I}_{p^2})$$

$$\quad \times\{\mathrm{vec}\mathbf{S}_{XX}^{-1}\otimes\{(\mathbf{D}_p\,'\otimes \mathbf{I}_{p^2})\mathrm{vec}\mathbf{S}_{XX}^{-1<2>}\}\}]/\partial \mathbf{s}_X\,'$$

$$= \partial[\{(\mathbf{D}_p\,'\otimes \mathbf{I}_{p^2})(\mathbf{I}_p\otimes \mathbf{K}_p\otimes \mathbf{I}_p)(\mathrm{vec}\mathbf{S}_{XX}^{-1}\otimes \mathrm{vec}\mathbf{S}_{XX}^{-1})\}$$

$$\quad \otimes \mathrm{vec}\mathbf{S}_{XX}^{-1} + (\mathbf{K}_{p^*,p^2}\otimes \mathbf{I}_{p^2})\{\mathrm{vec}\mathbf{S}_{XX}^{-1}\otimes\{(\mathbf{D}_p\,'\otimes \mathbf{I}_{p^2})$$

$$\quad \times(\mathbf{I}_p\otimes \mathbf{K}_p\otimes \mathbf{I}_p)(\mathrm{vec}\mathbf{S}_{XX}^{-1}\otimes \mathrm{vec}\mathbf{S}_{XX}^{-1})\}\}]/\partial \mathbf{s}_X\,'$$

(A1.5)

$$= -[(\mathbf{D}_p '\otimes \mathbf{I}_{p^2})(\mathbf{I}_p \otimes \mathbf{K}_p \otimes \mathbf{I}_p)\{(\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p) \otimes \mathrm{vec}\mathbf{S}_{XX}^{-1}$$

$$+ \mathrm{vec}\mathbf{S}_{XX}^{-1} \otimes (\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p)\}] \otimes \mathrm{vec}\mathbf{S}_{XX}^{-1}$$

$$-\{(\mathbf{D}_p '\otimes \mathbf{I}_{p^2})(\mathbf{I}_p \otimes \mathbf{K}_p \otimes \mathbf{I}_p)(\mathrm{vec}\mathbf{S}_{XX}^{-1})^{<2>}\} \otimes (\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p)$$

$$-(\mathbf{K}_{p^*,p^2} \otimes \mathbf{I}_{p^2})[(\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p) \otimes \{(\mathbf{D}_p '\otimes \mathbf{I}_{p^2})(\mathbf{I}_p \otimes \mathbf{K}_p \otimes \mathbf{I}_p)$$

$$\times (\mathrm{vec}\mathbf{S}_{XX}^{-1})^{<2>}\}]$$

$$-(\mathbf{K}_{p^*,p^2} \otimes \mathbf{I}_{p^2})[\mathrm{vec}\mathbf{S}_{XX}^{-1} \otimes \{(\mathbf{D}_p '\otimes \mathbf{I}_{p^2})(\mathbf{I}_p \otimes \mathbf{K}_p \otimes \mathbf{I}_p)$$

$$\times ((\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p) \otimes \mathrm{vec}\mathbf{S}_{XX}^{-1} + \mathrm{vec}\mathbf{S}_{XX}^{-1} \otimes (\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p))\}],$$

$$\partial \mathrm{vec}\left\{\partial \mathrm{vec}\left(\frac{\partial \hat{\boldsymbol{\beta}}}{\partial \mathbf{s}_X'}\right)\Big/ \partial \mathbf{s}_X'\right\}\Big/ \partial \mathbf{s}_{XY}' = \partial \mathrm{vec}\left\{\frac{\partial^2 \hat{\boldsymbol{\beta}}}{(\partial \mathbf{s}_X')^{<2>}}\right\}\Big/ \partial \mathbf{s}_{XY}'$$

$$= ([\{(\mathbf{D}_p '\mathbf{S}_{XX}^{-1<2>}) \otimes \mathrm{vec}'\mathbf{S}_{XX}^{-1} + \mathrm{vec}'\mathbf{S}_{XX}^{-1} \otimes (\mathbf{D}_p '\mathbf{S}_{XX}^{-1<2>})\}$$

$$\times (\mathbf{I}_p \otimes \mathbf{K}_p \otimes \mathbf{I}_p)] \otimes \mathbf{I}_{p^* \bullet p})\partial \mathrm{vec}(\mathbf{D}_p '\otimes \mathbf{s}_{XY}'\otimes \mathbf{I}_p)\Big/ \partial \mathbf{s}_{XY}', \qquad \text{(A1.6)}$$

where

$$\partial \mathrm{vec}(\mathbf{D}_p '\otimes \mathbf{s}_{XY}'\otimes \mathbf{I}_p)\Big/ \partial \mathbf{s}_{XY}'$$

$$= \partial (\mathbf{I}_{p^2} \otimes \mathbf{K}_{p^2,p^*} \otimes \mathbf{I}_p)\{\mathrm{vec}(\mathbf{D}_p ') \otimes \mathrm{vec}(\mathbf{s}_{XY}'\otimes \mathbf{I}_p)\}\Big/ \partial \mathbf{s}_{XY}'$$

$$= (\mathbf{I}_{p^2} \otimes \mathbf{K}_{p^2,p^*} \otimes \mathbf{I}_p)[\mathrm{vec}(\mathbf{D}_p ') \otimes \{(\partial \mathbf{s}_{XY}/\partial \mathbf{s}_{XY}') \otimes \mathrm{vec}\mathbf{I}_p\}] \qquad \text{(A1.7)}$$

$$= (\mathbf{I}_{p^2} \otimes \mathbf{K}_{p^2,p^*} \otimes \mathbf{I}_p)\{\mathrm{vec}(\mathbf{D}_p ') \otimes \mathbf{I}_p \otimes \mathrm{vec}\mathbf{I}_p\}.$$

1.2 The partial derivatives of the residual variance
(a) The first partial derivatives

$$\frac{\partial \hat{\psi}}{\partial \mathbf{s}_X} = \mathbf{D}_p '(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})^{<2>}, \quad \frac{\partial \hat{\psi}}{\partial \mathbf{s}_{XY}} = -2\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY}. \qquad \text{(A1.8)}$$

(b) The nonzero second partial derivatives

Using (A1.1),

$$\frac{\partial^2 \hat{\psi}}{\partial \mathbf{s}_X \partial \mathbf{s}_X'} = \mathbf{D}_p '[\{\partial \mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})/\partial \mathbf{s}_X'\} \otimes \mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})$$

$$+ \mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY}) \otimes \{\partial \mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})/\partial \mathbf{s}_X'\}]$$

$$= -\mathbf{D}_p '[(\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1}) \otimes \mathbf{S}_{XX}^{-1}\}\mathbf{D}_p) \otimes (\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY}) \qquad \text{(A1.9)}$$

$$+ (\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY}) \otimes (\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1}) \otimes \mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)].$$

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Similarly, we have

$$\frac{\partial^2 \hat{\psi}}{\partial \mathbf{s}_X \partial \mathbf{s}_{XY}{}'} = -2\frac{\partial \mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1}}{\partial \mathbf{s}_X} = 2\mathbf{D}_p{}'\{(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})\otimes\mathbf{S}_{XX}^{-1}\},$$

$$\frac{\partial^2 \hat{\psi}}{\partial \mathbf{s}_{XY}\partial \mathbf{s}_{XY}{}'} = -2\mathbf{S}_{XX}^{-1}. \tag{A1.10}$$

(c) The nonzero third partial derivatives

From (A1.9),

$$\partial\,\mathrm{vec}\left(\frac{\partial^2 \hat{\psi}}{\partial \mathbf{s}_X \partial \mathbf{s}_X{}'}\right)/\partial\mathbf{s}_X{}' = \partial\left\{\frac{\partial^2 \hat{\psi}}{(\partial\mathbf{s}_X)^{<2>}}\right\}/\partial\mathbf{s}_X{}'$$

$$= -(\mathbf{I}_{p*}\otimes\mathbf{D}_p{}')$$

$$\times[\ \{\partial\,\mathrm{vec}(\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)/\partial\mathbf{s}_X{}'\}\otimes\mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})$$

$$+\mathrm{vec}(\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)\otimes\{\partial\,\mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})/\partial\mathbf{s}_X{}'\}$$

$$+(K_{p*,p}\otimes\mathbf{I}_p)\{\ (\partial\,\mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})/\partial\mathbf{s}_X{}')$$

$$\otimes(\mathrm{vec}(\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p))$$

$$+\mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})\otimes(\partial\,\mathrm{vec}(\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)/\partial\mathbf{s}_X{}')\ \}\ ]. \tag{A1.11}$$

In (A1.11), let $\mathbf{U}*$ and $\mathbf{V}*$ be defined as follows:

$$\partial\mathrm{vec}(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})/\partial\mathbf{s}_X{}' = -\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p \equiv -\mathbf{U}* \tag{A1.12}$$

used before, and

$$\frac{\partial\,\mathrm{vec}(\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)}{\partial\mathbf{s}_X{}'}$$

$$= (\mathbf{D}_p{}'\otimes\mathbf{I}_p)\frac{\partial\,\mathrm{vec}\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}}{\partial\mathbf{s}_X{}'}$$

$$= -(\mathbf{D}_p{}'\otimes\mathbf{I}_p)[\ (\{(\mathbf{s}_{XY}{}'\mathbf{S}_{XX}^{-1})\otimes\mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)\otimes\mathrm{vec}\mathbf{S}_{XX}^{-1}$$

$$+(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})\otimes(\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p)\ ] \equiv -\mathbf{V}*. \tag{A1.13}$$

Using (A1.12) and (A1.13), (A1.11) becomes

$$\partial\,\mathrm{vec}\left(\frac{\partial^2 \hat{\psi}}{\partial\mathbf{s}_X\partial\mathbf{s}_X{}'}\right)/\partial\mathbf{s}_X{}'$$

$$= (\mathbf{I}_{p*}\otimes\mathbf{D}_p{}')[\ \mathbf{V}*\otimes(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})+(\mathrm{vec}\mathbf{U}*)\otimes\mathbf{U}*$$

$$+(K_{p*,p}\otimes\mathbf{I}_p)\{\mathbf{U}*\otimes\mathrm{vec}\mathbf{U}*+(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})\otimes\mathbf{V}*\}]. \tag{A1.14}$$

Similarly,

$$\partial \, \text{vec}\left(\frac{\partial^2 \hat{\psi}}{\partial \mathbf{s}_X \partial \mathbf{s}_X'}\right) / \partial \mathbf{s}_{XY}' = \partial \left\{\frac{\partial^2 \hat{\psi}}{(\partial \mathbf{s}_X)^{<2>}}\right\} / \partial \mathbf{s}_{XY}'$$

$$= -(\mathbf{I}_{p*} \otimes \mathbf{D}_p')$$

$$\times [\ \{\partial(\{\text{vec}(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1}) \otimes \mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)/\partial \mathbf{s}_{XY}'\} \otimes (\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})$$

$$+ \text{vec}(\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1}) \otimes \mathbf{S}_{XX}^{-1}\}\mathbf{D}_p) \otimes \{\partial(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})/\partial \mathbf{s}_{XY}'\} \qquad (A1.15)$$

$$+(K_{p*,p} \otimes \mathbf{I}_p)\{ \ (\partial(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})/\partial \mathbf{s}_{XY}') \otimes (\text{vec}(\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1}) \otimes \mathbf{S}_{XX}^{-1}\}\mathbf{D}_p))$$

$$+(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY}) \otimes (\partial \, \text{vec}(\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1}) \otimes \mathbf{S}_{XX}^{-1}\}\mathbf{D}_p)/\partial \mathbf{s}_{XY}') \ \} \ ].$$

Using

$$\partial \text{vec}[\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1}) \otimes \mathbf{S}_{XX}^{-1}\}\mathbf{D}_p]/\partial \mathbf{s}_{XY}'$$

$$= (\mathbf{D}_p'\otimes \mathbf{I}_p)\frac{\partial \text{vec}\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1}) \otimes \mathbf{S}_{XX}^{-1}\}}{\partial \mathbf{s}_{XY}'} \qquad (A1.16)$$

$$= (\mathbf{D}_p'\otimes \mathbf{I}_p)(\mathbf{S}_{XX}^{-1} \otimes \text{vec}\mathbf{S}_{XX}^{-1}),$$

(A1.15) becomes

$$\partial \, \text{vec}\left(\frac{\partial^2 \hat{\psi}}{\partial \mathbf{s}_X \partial \mathbf{s}_X'}\right) / \partial \mathbf{s}_{XY}'$$

$$= -(\mathbf{I}_{p*} \otimes \mathbf{D}_p')[\ \{(\mathbf{D}_p'\otimes \mathbf{I}_p)(\mathbf{S}_{XX}^{-1} \otimes \text{vec}\mathbf{S}_{XX}^{-1})\} \otimes (\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY})$$

$$+ \text{vec}(\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1}) \otimes \mathbf{S}_{XX}^{-1}\}\mathbf{D}_p) \otimes \mathbf{S}_{XX}^{-1}$$

$$+(K_{p*,p} \otimes \mathbf{I}_p)\{ \ \mathbf{S}_{XX}^{-1} \otimes \text{vec}(\{(\mathbf{s}_{XY}'\mathbf{S}_{XX}^{-1}) \otimes \mathbf{S}_{XX}^{-1}\}\mathbf{D}_p) \qquad (A1.17)$$

$$+(\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY}) \otimes ((\mathbf{D}_p'\otimes \mathbf{I}_p)(\mathbf{S}_{XX}^{-1} \otimes \text{vec}\mathbf{S}_{XX}^{-1})) \ \} \ ].$$

From (A1.10),

$$\partial \, \text{vec}\left(\frac{\partial^2 \hat{\psi}}{\partial \mathbf{s}_X \partial \mathbf{s}_{XY}'}\right) / \partial \mathbf{s}_{XY}'$$

$$= 2\{\mathbf{I}_p \otimes (\mathbf{D}_p'\mathbf{S}_{XX}^{-1<2>})\}(K_p \otimes \mathbf{I}_p)\left(\frac{\partial \mathbf{s}_{XY}}{\partial \mathbf{s}_{XY}'} \otimes \text{vec}\mathbf{I}_p\right) \qquad (A1.18)$$

$$= 2\{\mathbf{I}_p \otimes (\mathbf{D}_p'\mathbf{S}_{XX}^{-1<2>})\}(K_p \otimes \mathbf{I}_p)(\mathbf{I}_p \otimes \text{vec}\mathbf{I}_p),$$

which becomes, when the order of derivatives is changed,

$$\partial \,\mathrm{vec}\!\left(\frac{\partial^2 \hat{\psi}}{\partial \mathbf{s}_{XY} \partial \mathbf{s}_{XY}{}'}\right)\!\Big/\partial \mathbf{s}_X{}' = 2\mathbf{S}_{XX}^{-1<2>}\mathbf{D}_p. \tag{A1.19}$$

2. The direct proof of Theorem 1

(a) Asymptotic robustness of $\mathrm{avar}_{\mathrm{NT}}(\hat{\beta}_i)$

It is known that

$$(\mathbf{\Omega})_{ab,cd} = \kappa_{abcd} + (\mathbf{\Omega}_{\mathrm{NT}})_{ab,cd} = \kappa_{abcd} + \omega_{\mathrm{NT}ab,cd},$$
$$(p+1 \geq a \geq b \geq 1;\ p+1 \geq c \geq d \geq 1), \tag{A2.1}$$

(see e.g., Stuart & Ort, 1994, Equation (13.44)).   As was derived in (4.2), we have

$$n\,\mathrm{avar}(\hat{\beta}_i) = \sum_U \sum_V \frac{\partial \beta_i}{\partial \sigma_U}(\mathbf{\Omega})_{U,V}\frac{\partial \beta_i}{\partial \sigma_V},\ (i=1,..,p), \tag{A2.2}$$

where $\displaystyle\sum_U$ and $\displaystyle\sum_V$ denote the summations over the ranges:

$$U \in \{(a,b),\ (e,Y);\ p \geq a \geq b \geq 1,\ e=1,...,p\},$$
$$V \in \{(c,d),\ (g,Y);\ p \geq c \geq d \geq 1,\ g=1,...,p\},$$

respectively.   Using (A2.1), (A2.2) becomes

$$= -\sum_{a,b=1}^{p} \sum_V \beta_a \sigma^{bi}(\kappa_{abV} + \omega_{\mathrm{NT}ab,V})\frac{\partial \beta_i}{\partial \sigma_V} + \sum_{e=1}^{p} \sum_V \sigma^{ei}(\kappa_{YeV} + \omega_{\mathrm{NT}Ye,V})\frac{\partial \beta_i}{\partial \sigma_V}. \tag{A2.3}$$

In (A2.3), κ_{YeV} is the multivariate cumulant of Y, x_e and the two variables in V, say v_1 and v_2, which is equivalent to that of $\boldsymbol{\beta}'\mathbf{x}, x_e, v_1$ and v_2 due to the property of cumulants and the assumption of the independence of $\mathbf{x}$ and $\boldsymbol{\varepsilon}$. That is, using the notation $\mu(\cdot,...,\cdot)$ for the multivariate central moment of the argument variables,

$$\kappa_{YeV} = \mu\left(\sum_{k=1}^{p}\beta_k x_k, x_e, v_1, v_2\right) - \mu\left(\sum_{k=1}^{p}\beta_k x_k, x_e\right)\mu(v_1, v_2)$$

$$- \mu\left(\sum_{k=1}^{p}\beta_k x_k, v_1\right)\mu(x_e, v_2) - \mu\left(\sum_{k=1}^{p}\beta_k x_k, v_2\right)\mu(x_e, v_1)$$

$$= \sum_{k=1}^{p}\beta_k\{\mu(x_k, x_e, v_1, v_2) - \mu(x_k, x_e)\mu(v_1, v_2)$$

$$- \mu(x_k, v_1)\mu(x_e, v_2) - \mu(x_k, v_2)\mu(x_e, v_1)\}$$

$$= \sum_{k=1}^{p}\beta_k\kappa_{keV}. \tag{A2.4}$$

Using (A2.3) and (A2.4), we have

$$n \, \mathrm{avar}(\hat{\beta}_i) = -\sum_{a,b=1}^{p}\sum_{V}\beta_a\sigma^{bi}\omega_{\mathrm{NT}ab,V}\frac{\partial\beta_i}{\partial\sigma_V} + \sum_{e=1}^{p}\sum_{V}\sigma^{ei}\omega_{\mathrm{NT}Ye,V}\frac{\partial\beta_i}{\partial\sigma_V}$$

$$= n \, \mathrm{avar}_{\mathrm{NT}}(\hat{\beta}_i), \quad (i=1,...,p). \tag{A2.5}$$

(b) Asymptotic robustness of $\mathrm{abis}_{\mathrm{NT}}(\hat{\beta}_i)$

In a similar manner, we have

$$n \, \mathrm{abis}(\hat{\beta}_i) = \frac{1}{2}\sum_{U}\sum_{V}\frac{\partial^2\beta_i}{\partial\sigma_U\partial\sigma_V}(\mathbf{\Omega})_{U,V}$$

$$= \frac{1}{2}\sum_{p\geq a\geq b\geq 1}\sum_{p\geq c\geq d\geq 1}\frac{\partial^2\beta_i}{\partial\sigma_{ab}\partial\sigma_{cd}}(\mathbf{\Omega})_{ab,cd} + \sum_{p\geq c\geq d\geq 1}\sum_{e=1}^{p}\frac{\partial^2\beta_i}{\partial\sigma_{ab}\partial\sigma_{Ye}}(\mathbf{\Omega})_{ab,Ye}$$

$$= \sum_{a,b,c,d=1}^{p}\beta_c\sigma^{da}\sigma^{bi}(\kappa_{abcd} + \omega_{\mathrm{NT}ab,cd}) - \sum_{a,b,e=1}^{p}\sigma^{ea}\sigma^{bi}(\kappa_{abeY} + \omega_{\mathrm{NT}ab,eY}), \tag{A2.6}$$

$$(i=1,...,p).$$

Since $\kappa_{abeY} = \sum_{k=1}^{p}\beta_k\kappa_{abek}$ (see (A2.4)), (A2.6) becomes

$$= \sum_{a,b,c,d=1}^{p}\beta_c\sigma^{da}\sigma^{bi}\omega_{\mathrm{NT}ab,cd} - \sum_{a,b,e=1}^{p}\sigma^{ea}\sigma^{bi}\omega_{\mathrm{NT}ab,eY}$$

$$= n \, \mathrm{abis}_{\mathrm{NT}}(\hat{\beta}_i) = 0.$$

(c) Asymptotic robustness of $\mathrm{abis}_{\mathrm{NT}}(\hat{\psi})$

Noting that s_{YY} in $\hat{\psi}$ does not contribute to the asymptotic/exact bias of

$\hat{\psi}$, we can use the same ranges of U and V as before:

$$n\,\mathrm{abis}(\hat{\psi}) = \frac{1}{2}\sum_{U}\sum_{V}\frac{\partial^2\psi}{\partial\sigma_U\partial\sigma_V}(\boldsymbol{\Omega})_{U,V}$$

$$= \frac{1}{2}\sum_{p\geq a\geq b\geq 1}\sum_{p\geq c\geq d\geq 1}\frac{\partial^2\psi}{\partial\sigma_{ab}\partial\sigma_{cd}}(\boldsymbol{\Omega})_{ab,cd} + \sum_{p\geq c\geq d\geq 1}\sum_{e=1}^{p}\frac{\partial^2\psi}{\partial\sigma_{ab}\partial\sigma_{Ye}}(\boldsymbol{\Omega})_{ab,Ye}$$

$$+ \frac{1}{2}\sum_{e,f=1}^{p}\frac{\partial^2\psi}{\partial\sigma_{Ye}\partial\sigma_{Yf}}(\boldsymbol{\Omega})_{Ye,Yf}$$

$$= -\sum_{a,b,c,d=1}^{p}\beta_c\sigma^{da}\beta_b(\kappa_{abcd}+\omega_{\mathrm{NT}ab,cd}) \qquad\qquad (\text{A2.7})$$

$$+2\sum_{a,b,e=1}^{p}\sigma^{ea}\beta_b(\kappa_{abYe}+\omega_{\mathrm{NT}ab,Ye}) - \sum_{e,f=1}^{p}\sigma^{ef}(\kappa_{YeYf}+\omega_{\mathrm{NT}Ye,Yf})$$

$$= n\,\mathrm{abis}_{\mathrm{NT}}(\hat{\psi}),$$

where $\kappa_{YeYf} = \sum_{l,m=1}^{p}\beta_l\kappa_{lemf}\beta_m$ is used. Q. E. D.

References

Magnus, J. R., & Neudecker, H. (1999). *Matrix differential calculus with applications in statistics and econometrics* (Rev. ed.). New York: Wiley.

Ogasawara, H. (2007). Asymptotic expansion and asymptotic robustness of the normal-theory estimators in the random regression model. *Journal of Statistical Computation and Simulation*, 77, 821-838.

Stuart, A., & Ord, J. K. (1994). *Kendall's advanced theory of statistics: Distribution theory* (6th ed., Vol.1). London: Arnold.