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Supplement to the paper "Maximization of some types of information for unidentified item response models with guessing parameters"

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This article supplements Ogasawara (2021).

Reference

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In the following, the number of distinct θ_j 's among $\theta_j\,(j=1,...,N)$ is assumed to be sufficiently large with the largest one being *N*. As addressed in Ogasawara (2021), in the case of the 1PL-G model, k_2 is associated with the location indeterminacies of $a^*\theta_j^*$ and $a^*b_i^*$. Consequently, under $\bar{\theta}=\bar{\theta}^*=k_\theta$, k_2 can be set to 1. Define $\mathrm{var}\{\ln(e^{a\theta}+k_1)\}$ as the variance of $\ln\{\exp(a\theta_j)+k_1\}$ $(j=1,...,N)$. Let

$$\theta_{\min}\equiv\min\{\theta_j;\ j=1,...,N\}\ \text{ with }\ \text{inf-}k_1\equiv-\exp(a\theta_{\min}). \qquad \text{(a.1)}$$

Then, we have the following result.

Lemma 1. *In the case of the 1PL-G model*,

$$\lim_{k_1\to\text{inf-}k_1+0}\mathrm{var}\{\ln(e^{a\theta}+k_1)\}=+\infty\,. \qquad \text{(a.2)}$$

Proof. Let $K_j\equiv\exp(a\theta_j)+k_1\ (j=1,..,N)$ and $K_{\min}\equiv\exp(a\theta_{\min})+k_1$. Then,

$$\operatorname{var}\{\ln(e^{a\theta}+k_1)\} = N^{-1}\sum_{j=1}^{N}\left[\ln\{\exp(a\theta_j)+k_1\} - \overline{\ln(e^{a\theta}+k_1)}\right]^2$$

$$= N^{-1}\sum_{j=1}^{N}\left(\ln K_j - N^{-1}\sum_{m=1}^{N}\ln K_m\right)^2 > N^{-1}\left(\ln K_{\min} - N^{-1}\sum_{m=1}^{N}\ln K_m\right)^2 \tag{a.3}$$

$$= N^{-1}\left\{(1-N^{-1})\ln K_{\min} - N^{-1}\sum_{m=1\,(m\neq\min)}^{N}\ln K_m\right\}^2.$$

When $k_1 \to \text{inf-}k_1 + 0$, by definition $\ln K_{\min} \to -\infty$. Then, since $-N^{-1}\sum_{m=1\,(m\neq\min)}^{N}\ln K_m$ is finite, the last result in (a.3) goes to $+\infty$ Q.E.D.

A.1 The results under $a^* = [\operatorname{var}\{\ln(e^{a\theta}+k_1)\}]^{1/2}$

In this section the results under $a^* = [\operatorname{var}\{\ln(e^{a\theta}+k_1)\}]^{1/2}$ with $\overline{\theta} = \overline{\theta}^* = 0$ and $\operatorname{var}(\theta) = \operatorname{var}(\theta^*) = 1$ are shown.

Theorem 2. *Under* $a^* = [\operatorname{var}\{\ln(e^{a\theta}+k_1)\}]^{1/2}$ *in the 1PL-G model*,

$$\lim_{k_1\to\text{inf-}k_1+0} a^* = +\infty, \quad \lim_{k_1\to\text{inf-}k_1+0} b_i^* = \frac{N^{1/2}}{N-1},$$

$$0 < \lim_{k_1\to\text{inf-}k_1+0} c_i^* = \frac{c_i\exp(ab_i) - \text{inf-}k_1}{\exp(ab_i) - \text{inf-}k_1} < 1, \tag{a.4}$$

$$\lim_{k_1\to\text{inf-}k_1+0} \theta_{\min}^* = -N^{1/2} \quad \text{with} \quad \theta_{\min}^* = \ln\{\exp(a\theta_{\min}) + k_1\}$$

$$\lim_{k_1\to\text{inf-}k_1+0} \theta_j^* = \lim_{k_1\to\text{inf-}k_1+0} b_i^* = \frac{N^{1/2}}{N-1} \; (i=1,\ldots,n;\ j=1,..,N;\ j\neq\min).$$

Proof. $\lim_{k_1\to\text{inf-}k_1+0} a^* = +\infty$ is given by Lemma 1. For b_i^*, let $K_j^* = 1/K_j = 1/\{\exp(a\theta_j)+k_1\}\ (j=1,\ldots,N)$ and $K_{\min}^* = 1/K_{\min}$. Denote $\operatorname{var}\{\ln(e^{a\theta}+k_1)\} = \operatorname{var}[\ln\{1/(e^{a\theta}+k_1)\}]$ by $\operatorname{var}(\ln K^*)$. When $k_1 \to \text{inf-}k_1 + 0$, we find from Lemma 1 that the denominator of b_i^* in the first paragraph of Section 4 i.e., $\{\operatorname{var}(\ln K^*)\}^{1/2} \to +\infty$. On the other hand, for the numerator of b_i^*, when $k_1 \to \text{inf-}k_1 + 0$, using $\ln K_{\min} \to -\infty$ and $\ln K_{\min}^* \to +\infty$, we have

$$\lim_{k_1 \to \text{inf-}k_1+0} [\ln\{\exp(ab_i) - k_1\} - \overline{\ln(e^{a\theta} + k_1)}]$$

$$= \ln\{\exp(ab_i) - \text{inf-}k_1\} - N^{-1} \sum_{j=1(j \neq \min)}^{N} \ln\{\exp(a\theta_j) + \text{inf-}k_1\}$$

$$+ N^{-1} \lim_{K_{\min}^* \to +\infty} \ln K_{\min}^* \tag{a.5}$$

$$= N^{-1} \lim_{K_{\min}^* \to +\infty} \ln K_{\min}^* = +\infty.$$

Then,

$$\lim_{K_{\min}^* \to +\infty} b_i^* = \lim_{K_{\min}^* \to +\infty} \frac{N^{-1} \ln K_{\min}^*}{\{\text{var}(\ln K^*)\}^{1/2}}$$

$$= \lim_{K_{\min}^* \to +\infty} N^{-1/2} \left\{ \sum_{j=1}^{N} \left(\frac{\ln K_j^*}{\ln K_{\min}^*} - N^{-1} \sum_{m=1}^{N} \frac{\ln K_m^*}{\ln K_{\min}^*} \right)^2 \right\}^{-1/2} = \frac{N^{-1/2}}{1 - N^{-1}} = \frac{N^{1/2}}{N - 1} \tag{a.6}$$

and the results for c_i^* are obvious $(i = 1, ..., n)$.

For $\theta_{\min}^* = \ln\{\exp(a\theta_{\min}) + k_1\}$, we have

$$\lim_{k_1 \to \text{inf-}k_1+0} \theta_{\min}^* = \lim_{k_1 \to \text{inf-}k_1+0} \frac{\ln\{\exp(a\theta_{\min} + k_1)\} - \overline{\ln(e^{a\theta} + k_1)}}{[\text{var}\{\ln(e^{a\theta} + k_1)\}]^{1/2}}$$

$$= \lim_{K_{\min}^* \to +\infty} \frac{-\ln K_{\min}^* + \overline{\ln K^*}}{\{\text{var}(\ln K^*)\}^{1/2}} = -\frac{1 - N^{-1}}{N^{-1/2}(1 - N^{-1})} = -N^{1/2}. \tag{a.7}$$

For $\theta_j^* (j = 1, ..., N; j \neq \min)$, as for b_i^*, we obtain

$$\lim_{k_1 \to \text{inf-}k_1+0} \theta_j^* = \lim_{k_1 \to \text{inf-}k_1+0} \frac{\ln\{\exp(a\theta_j + k_1)\} - \overline{\ln(e^{a\theta} + k_1)}}{[\text{var}\{\ln(e^{a\theta} + k_1)\}]^{1/2}}$$

$$= \lim_{K_{\min}^* \to +\infty} \frac{N^{-1} \ln K_{\min}^*}{\{\text{var}(\ln K^*)\}^{1/2}} = \lim_{k_1 \to \text{inf-}k_1+0} b_i^* \; (i = 1, ..., n) = \frac{N^{1/2}}{N - 1}. \text{ Q.E.D.} \tag{a.8}$$

It is easily confirmed that

$$\overline{\lim_{k_1 \to \text{inf-}k_1+0} \theta^*} \equiv N^{-1} \sum_{j=1}^{N} \lim_{k_1 \to \text{inf-}k_1+0} \theta_j^* = 0 . \quad \text{(a.9)}$$

However,

$$\text{var}\left(\lim_{k_1 \to \text{inf-}k_1+0} \theta^* \right) \equiv N^{-1} \sum_{j=1}^{N} \left(\lim_{k_1 \to \text{inf-}k_1+0} \theta_j^* - \overline{\lim_{k_1 \to \text{inf-}k_1+0} \theta^*} \right)^2 = \frac{N}{N-1} > 1 . \quad \text{(a.10)}$$

When $k_1 \to \text{inf-}k_1 + 0$, $\Psi_{i\min}^* \ (\equiv \Psi_{ij}^* = 1/[1 + \exp\{-a^*(\theta_j^* - b_i^*)\}]$ when $\theta_j^* = \theta_{\min}^* = \ln\{\exp(a\theta_{\min}) + k_1\}$) goes to zero, and consequently, $P_{i\min} \ (\equiv P_{ij}$ when $\theta_j = \theta_{\min}$ or equivalently $\theta_j^* = \theta_{\min}^*$) goes to c_i^*. The last result holds only for $\theta_{\min}^*$ since $-a^*(\theta_j^* - b_i^*) = \ln\{\exp(ab_i) - k_1\} - \ln\{\exp(a\theta_j) + k_1\}$ is finite for $\theta_j \ (j = 1, ..., N; j \neq \min)$.

Lemma 2. *Under* $a^* = [\text{var}\{\ln(e^{a\theta} + k_1)\}]^{1/2}$ *in the 1PL-G model*,

$$\lim_{k_1 \to \text{inf-}k_1+0} \frac{\partial P_i^*}{\partial \theta^*} |_{\theta^* = \theta_{\min}^*} \equiv \lim_{k_1 \to \text{inf-}k_1+0} \frac{\partial P_i^*}{\partial \theta_{\min}^*} = 0 \quad (i = 1, ..., n) . \quad \text{(a.11)}$$

Proof. Recall that $K_j^* = 1/K_j = 1/\ln\{\exp(a\theta_j) + k_1\} \ (j = 1, ..., N)$ and $K_{\min}^* = 1/K_{\min}$. Then, as derived in Section 3 we have

$$\frac{\partial P_i^*}{\partial \theta_{\min}^*} = \frac{\{\text{var}(\ln K^*)\}^{1/2}\{\exp(a\theta_{\min}) + k_1\}(1 - P_{i\min})}{\exp(a\theta_{\min}) + \exp(ab_i)}$$

$$\equiv \frac{\{\text{var}(\ln K^*)\}^{1/2}}{K_{\min}^*} h_i \quad (i = 1, ..., n), \quad \text{(a.12)}$$

where $h_i = (1 - P_{i\min}) / \{\exp(a\theta_{\min}) + \exp(ab_i)\}$ does not depend on k_1; and $\text{var}(\ln K^*) = \text{var}[\ln\{1/(e^{a\theta} + k_1)\}] = \text{var}\{\ln(e^{a\theta} + k_1)\}$.

When $k_1 \to \text{inf-}k_1 + 0$, we have $\ln K_{\min}^* \to +\infty$ and from Lemma 1 $\text{var}(\ln K^*) \to +\infty$. Using L'Hôpital's rule, we obtain

$$\lim_{k_1 \to \text{inf-}k_1+0} \frac{\partial P_i^*}{\partial \theta_{\min}^*} = \lim_{K_{\min}^* \to +\infty} \frac{\partial \{\text{var}(\ln K^*)\}^{1/2} / \partial K_{\min}^*}{\partial K_{\min}^* / \partial K_{\min}^*} h_i$$

$$= \frac{1}{2}\{\text{var}(\ln K^*)\}^{-1/2} 2 \lim_{K_{\min}^* \to +\infty} \left(N^{-1} \frac{\ln K_{\min}^*}{K_{\min}^*} - N^{-2} \sum_{j=1}^{N} \frac{\ln K_j^*}{K_{\min}^*} \right) h_i \qquad \text{(a.13)}$$

$$= 0 \quad (i = 1,\ldots,n),$$

where $\lim_{K_{\min}^* \to +\infty} (\ln K_{\min}^*) / K_{\min}^* = 0$ is given again by L'Hôpital's rule. Q.E.D.

Then, we obtain the following main result.

Theorem 3. *Under* $a^* = [\text{var}\{\ln(e^{a\theta} + k_1)\}]^{1/2}$ *in the 1PL-G model,*

$$\lim_{k_1 \to \text{inf-}k_1+0} \sum_{i=1}^{n} I_{\text{F}i\min*} = \lim_{k_1 \to \text{inf-}k_1+0} I_{\text{S}\min*} = \lim_{k_1 \to \text{inf-}k_1+0} I_{\text{Q}\min*} = 0 \quad \text{and} \qquad \text{(a.14)}$$

$$\lim_{k_1 \to \text{inf-}k_1+0} I_{\text{F*}}^{+} = \lim_{k_1 \to \text{inf-}k_1+0} I_{\text{S*}}^{+} = \lim_{k_1 \to \text{inf-}k_1+0} I_{\text{Q*}}^{+} = +\infty\,, \qquad \text{(a.15)}$$

where $I_{\text{S}\min*} = I_{\text{S}j*}$ *when* $\theta_j = \theta_{\min}$ *with other similar expressions defined similarly.*

On the other hand, when $k_1 \to \text{sup-}k_1 - 0$, *all the values of* $I_{\text{F*}}^{+}, I_{\text{S*}}^{+}$ *and* $I_{\text{Q*}}^{+}$ *are finite and their unattained limiting values are given by* $k_1 = \text{sup-}k_1$ *in* $\partial P_i^* / \partial \theta_j^*$ $(i = 1,\ldots,n; j = 1,\ldots,N)$ *of the total informations, and* $c_{\text{sup-}k_1}^*$ ($\equiv c_i^*$ *when* $b_i = \min\{b_m; m = 1,\ldots,n\}$) *goes to* $-\infty$.

Proof. The first set of limiting zero informations (see (a.14)) is given by Lemma 2. For the second set of their infinite limiting values (see (a.15)), when $k_1 \to \text{inf-}k_1 + 0$, it is found that

$$\frac{\partial P_i^*}{\partial \theta_j^*} = [\text{var}\{\ln(e^{a\theta} + k_1)\}]^{1/2} \{\exp(a\theta_j) + k_1\} h_i$$
$$(i = 1,\ldots,n; j = 1,\ldots,N; j \neq \min) \qquad \text{(a.16)}$$

go to $+\infty$ since $\text{var}\{\ln(e^{a\theta} + k_1)\} \to +\infty$ and $\exp(a\theta_j) + k_1$ is finite as h_i, which gives the second set of infinite limiting informations.

The results when $k_1 \to \text{sup-}k_1 - 0$ are obviously derived since all the factors in $\partial P_i^* / \partial \theta_j^*$ are finite for this limiting case while

$c_i^* = \{c_i \exp(ab_i) - k_1\} / \{\exp(ab_i) - k_1\}$, when $c_i^* = c^*_{\text{sup-}k_1}$ goes to $-\infty$ since the numerator is negative and finite and the denominator approaches +0. Q.E.D.

A.2 The results under $a = a^* = k_3\,(>0)$

Next, we consider the case of parametrization with $a = a^* = k_3\,(>0)$, where $k_3 = 1$ is used without loss of generality. That is, ab_i and $a\theta_j$ are redefined as b_i and θ_j, respectively before transformation with $\bar{\theta} = N^{-1}\sum_{j=1}^{N}\theta_j = 0$ to remove the location indeterminacy. After transformation, using $a^* = 1$ we have

$$b_i^* = \ln\{\exp(b_i) - k_1\} - \overline{\ln(e^{\theta} + k_1)},\ \ c_i^* = \frac{c_i \exp(b_i) - k_1}{\exp(b_i) - k_1} \tag{a.17}$$

$$\theta_j^* = \ln\{\exp(\theta_j) + k_1\} - \overline{\ln(e^{\theta} + k_1)} \ \text{ with } \ \bar{\theta}^* = N^{-1}\sum_{m=1}^{N}\theta_m^* = 0$$

$(i = 1,...,n; j = 1,...,N)$.

We have two possible regions of k_1 as given in Section 2:

$$\text{inf-}k_1 = -\min\{\exp(\theta_j);\ j = 1,...,N\} < k_1 < \min\{\exp(b_i);\ i = 1,...,n\} = \text{sup-}k_1, \tag{a.18}$$

and $$\text{inf-}k_1 < k_1 \le \min\{c_i \exp(b_i);\ i = 1,...,n\} = \text{max-}k_1 < \text{sup-}k_1. \tag{a.19}$$

Define $\theta_{\min} = \min\{\theta_j; j = 1,...,N\}$ as before with similar expressions defined similarly. Then, we have the following results.

Theorem 4. *Under* $a = a^* = 1$ *and* $\bar{\theta} = \bar{\theta}^* = 0$ *in the 1PL-G model*,

$$\lim_{k_1 \to \text{inf-}k_1 + 0} b_i^* = +\infty,\ \ \lim_{k_1 \to \text{inf-}k_1 + 0} c_i^* = \frac{c_i \exp(b_i) - \text{inf-}k_1}{\exp(b_i) - \text{inf-}k_1}\ (<1) \ \text{ is finite,}$$

$$\lim_{k_1 \to \text{sup-}k_1 - 0} c^*_{\text{sup-}k_1} = -\infty,\ \ \lim_{k_1 \to \text{sup-}k_1 - 0} c^*_{i\,(i \ne \text{sup-}k_1)} \ \text{ is finite,} \tag{a.20}$$

$$\lim_{k_1 \to \text{inf-}k_1 + 0} \theta^*_{\min} = -\infty,\ \ \lim_{k_1 \to \text{inf-}k_1 + 0} \theta^*_{j(j \ne \min)} = +\infty \ \text{ with } \ \overline{\lim_{k_1 \to \text{inf-}k_1 + 0} \theta^*} = 0 \ \text{ and}$$

$$\text{var}\left(\lim_{k_1 \to \text{inf-}k_1 + 0} \theta^*\right) = +\infty \ \ (i = 1,...,n; j = 1,...,N).$$

Proof. The results are given as in Lemma 1 and Theorem 2 with

$a = a^* = 1$ and $\overline{\theta} = \overline{\theta}^* = 0$. Q.E.D.

Lemma 3. *Under* $a = a^* = 1$ *and* $\overline{\theta} = \overline{\theta}^* = 0$ *in the 1PL-G model,*

$$\lim_{k_1 \to \text{inf-}k_1 + 0} \partial P_i^* / \partial \theta_{\min}^* = 0 \quad \text{and} \quad \lim_{k_1 \to \text{inf-}k_1 + 0} \partial P_i^* / \partial \theta_j^* \tag{a.21}$$

$(i = 1,...,n;\ j = 1,...,N; j \neq \min)$ *are positive and finite*.

Proof. The zero limiting value is given by $\lim_{k_1 \to \text{inf-}k_1 + 0} \partial P_i^* / \partial \theta_{\min}^* =$

$\lim_{k_1 \to \text{inf-}k_1 + 0} \dfrac{\{\exp(\theta_{\min}) + k_1\}(1 - P_{ij})}{\exp(\theta_{\min}) + \exp(b_i)} = 0$ since $\exp(\theta_{\min}) + k_1 \to +0$. On the other hand,

$$\lim_{k_1 \to \text{inf-}k_1 + 0} \frac{\partial P_i^*}{\partial \theta_j^*} = \lim_{k_1 \to \text{inf-}k_1 + 0} \frac{\{\exp(\theta_j) + k_1\}(1 - P_{ij})}{\exp(\theta_j) + \exp(b_i)}$$

$$= \frac{\{\exp(\theta_j) + \text{inf-}k_1\}(1 - P_{ij})}{\exp(\theta_j) + \exp(b_i)} (i = 1,...,n;\ j = 1,...,N; j \neq \min), \tag{a.22}$$

which are obviously positive and finite by definition. Q.E.D.

Theorem 5. *Under* $a = a^* = 1$ *and* $\overline{\theta} = \overline{\theta}^* = 0$ *in the 1PL-G model,*

$$\lim_{k_1 \to \text{inf-}k_1 + 0} \sum_{i=1}^{n} I_{\text{F}i\min *} = \lim_{k_1 \to \text{inf-}k_1 + 0} I_{\text{Smin}*} = \lim_{k_1 \to \text{inf-}k_1 + 0} I_{\text{Qmin}*} = 0 \text{ ; and} \tag{a.23}$$

$$\lim_{k_1 \to \text{inf-}k_1 + 0} I_{\text{F}*}^{+} = \sum_{j=1}^{N} \sum_{i=1}^{n} I_{\text{F}ij*},\ \lim_{k_1 \to \text{inf-}k_1 + 0} I_{\text{S}*}^{+} = \sum_{j=1}^{N} I_{\text{S}j*} \text{ and } \lim_{k_1 \to \text{inf-}k_1 + 0} I_{\text{Q}*}^{+} = \sum_{j=1}^{N} I_{Qj*} \tag{a.24}$$

are finite, where the right-hand side in each equation of (a.24) is defined to be given by $k_1 = \text{inf-}k_1$.

When $k_1 \to \text{sup-}k_1 - 0$, *all the values of* $I_{\text{F}*}^{+}, I_{\text{S}*}^{+}$ *and* $I_{\text{Q}*}^{+}$ *are finite and their unattained limiting values are given by* $k_1 = \text{sup-}k_1$ *in* $\partial P_i^* / \partial \theta_j^*$ $(i = 1,...,n; j = 1,...,N)$ *of the total informations, and* $c_{\text{sup-}k_1}^*$ $(\equiv c_i^*$ *when* $b_i = \min\{b_m; m = 1,...,n\}$ *) goes to* $-\infty$.

Proof. Using Lemma 3 and the definitions of the informations, (a.23) and (a.24) follow. The results when $k_1 \to \text{sup-}k_1 - 0$ are given as in Theorem 3. Q.E.D.

Recall that under $a^* = [\text{var}\{\ln(e^{a\theta} + k_1)\}]^{1/2}$, $\text{var}\left(\lim_{k_1 \to \text{inf-}k_1+0} \theta^*\right)$ is finite while $I^+_{\text{F*}}, I^+_{\text{S*}}$ and $I^+_{\text{Q*}}$ go to $+\infty$ when $k_1 \to \text{inf-}k_1 + 0$. To the contrary, under $a = a^* = 1$, the opposite results with infinite $\text{var}\left(\lim_{k_1 \to \text{inf-}k_1+0} \theta^*\right)$ and finite $I^+_{\text{F*}}, I^+_{\text{S*}}$ and $I^+_{\text{Q*}}$ when $k_1 \to \text{inf-}k_1 + 0$ are obtained.

Theorem 6. *Under* $a = a^* = 1$ *and* $\bar{\theta} = \bar{\theta}^* = 0$ *in the 1PL-G model, using the possible region of (a.18) for* k_1*, the total informations* $I^+_{\text{F*}}, I^+_{\text{S*}}$ *and* $I^+_{\text{Q*}}$ *have no maxima though their suprema are finite, which are given when* $k_1 = \text{sup-}k_1$*. When the possible region of (a.19) for* k_1 *is used, the total informations have finite maxima, which are obtained by* $k_1 = \text{max-}k_1$.

Proof. Since $\frac{\partial P_i^*}{\partial \theta_j^*} = \frac{\{\exp(\theta_j) + k_1\}(1 - P_{ij})}{\exp(\theta_j) + \exp(b_i)}$ $(i = 1,...,n;\ j = 1,...,N)$, the total informations are increasing functions of k_1, which gives the results depending on the domains of definition for k_1. Q.E.D.

Corollary 2. *Under the same condition as in Theorem 6 using* $\text{max-}k_1$ *in (a.19) for* k_1*, when* $c_i = 0$ *for at least one item, the maxima of the informations are already attained before transformation*.

Proof. When $c_i = 0$ for an item, $\text{max-}k_1 = \min\{c_m \exp(b_m);\ m = 1,...,n\}$ becomes 0, which gives the required result. Q.E.D.

Corollary 2 shows a flexibility of the model with negative c_i^*. Even when $c_i = 0$ for all items, the informations can further be increased. Note that in this case the model before transformation is the usual 1-parameter logistic or Rasch model.