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Supplement to the paper "On some known derivations and new ones for the Wishart distribution: A didactic"

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This article supplements Ogasawara (2023) with the second proof and associated remarks for Lemma 1.

Lemma 1 (Ogasawara, 2023). *Suppose that each of* $2m$ *variables* X_{ik} *and* X_{jk} $(i \ne j; k = 1,...,m;\ m = 1,2,...)$ *independently follows* $\mathrm{N}(0,1) \equiv \mathrm{N}_1(0,1)$. *Then, the distribution of* $\sum_{k=1}^{m} X_{ik} X_{jk}$ *is the same as that of* $X_{il}\sqrt{\sum_{k=1}^{m} X_{jk}^2}$ $(i \ne j;\ l = 1,...,m)$.

Proof 2. In this proof, the pdf of the chi-distribution is used with associated mgf's. Let $X = \sqrt{\sum_{k=1}^{m} X_{jk}^2}$ and $Y = X_{il}$. Then, X is chi-distributed with m df. The pdf of X at x denoted by $f_\chi(x \mid m)$ is given by that of the chi-square distributed $U = X^2$ at u with m df i.e., $f_{\chi^2}(u \mid m) = \dfrac{u^{(m/2)-1}}{2^{m/2}\Gamma(m/2)}\exp(-u/2)$ with the Jacobian $\mathrm{d}u / \mathrm{d}x = 2x$, yielding

$$\begin{aligned} f_\chi(x \mid m) &= \frac{u^{(m/2)-1}}{2^{m/2}\Gamma(m/2)}\exp(-u/2)\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= \frac{u^{(m/2)-1}\exp(-u/2)}{2^{m/2}\Gamma(m/2)}2x \\ &= \frac{x^{m-1}\exp(-x^2/2)}{2^{(m/2)-1}\Gamma(m/2)}. \end{aligned}$$

Then, the joint pdf of X and Y becomes $\dfrac{x^{m-1}\exp(-x^2/2)}{2^{(m/2)-1}\Gamma(m/2)}\dfrac{\exp(-y^2/2)}{\sqrt{2\pi}}$.

Consider the variable transformation $Z = XY$ with unchanged X. Since the Jacobian is $J(Y \to Z) = x^{-1}$, the joint pdf of X and Z is

$$\frac{x^{m-1}\exp(-x^2/2)}{2^{(m/2)-1}\Gamma(m/2)}\frac{\exp(-y^2/2)}{\sqrt{2\pi}}x^{-1} = \frac{x^{m-2}\exp\{-(x^2+z^2x^{-2})/2\}}{2^{(m-1)/2}\pi^{1/2}\Gamma(m/2)},$$

which gives the pdf of Z as $f_Z(z\,|\,m) = \dfrac{\int_0^\infty x^{m-2}\exp\{-(x^2+z^2x^{-2})/2\}\,\mathrm{d}\,x}{2^{(m-1)/2}\pi^{1/2}\Gamma(m/2)}$. The mgf of Z is

$$\frac{\int_{-\infty}^\infty\int_0^\infty x^{m-2}\exp\{-(x^2+z^2x^{-2})/2\}\exp(tz)\,\mathrm{d}\,x\,\mathrm{d}\,z}{2^{(m-1)/2}\pi^{1/2}\Gamma(m/2)}$$

$$= \frac{\int_0^\infty x^{m-2}\exp\{-(1-t^2)x^2/2\}\int_{-\infty}^\infty \exp\{-(z-tx^2)^2x^{-2}/2\}\,\mathrm{d}\,z\,\mathrm{d}\,x}{2^{(m-1)/2}\pi^{1/2}\Gamma(m/2)}$$

$$= \frac{\int_0^\infty x^{m-2}\exp\{-(1-t^2)x^2/2\}x(2\pi)^{1/2}\,\mathrm{d}\,x}{2^{(m-1)/2}\pi^{1/2}\Gamma(m/2)}$$

$$= \frac{\int_0^\infty (1-t^2)^{m/2}x^{m-1}\exp\{-(1-t^2)x^2/2\}\{2^{(m/2)-1}\Gamma(m/2)\}^{-1}\,\mathrm{d}\,x}{(1-t^2)^{m/2}}$$

$$= \frac{1}{(1-t^2)^{m/2}},$$

where the integrand of the last integral is the density of the scaled chi-distributed variable with the scale parameter $(1-t^2)^{-1/2}$ $(t^2 < 1)$.

Noting that the distribution of each $X_{ik}X_{jk}$ in $\sum_{k=1}^m X_{ik}X_{jk}$ is equal to that of $|X_{ik}|X_{jk}$, which is distributed as Z given earlier when *m* = 1, the mgf of $X_{ik}X_{jk}$ becomes $(1-t^2)^{-/2}$. Since the mgf of $\sum_{k=1}^m X_{ik}X_{jk}$ is equal to that of $\sum_{k=1}^m |X_{ik}|X_{jk}$ with the *m* terms being i.i.d., the mgf of $\sum_{k=1}^m X_{ik}X_{jk}$ is given by $(1-t^2)^{-m/2}$ as obtained earlier for $\sqrt{\sum_{k=1}^m X_{jk}^2}\,Y_{il}$ showing their same distributions. Q.E.D.

Remark S.1 A byproduct of Proof 2 is the pdf of Z using an integral expression. A slightly different derivation of the pdf is given by the variable transformation $Z = XY$ with unchanged Y rather than X. Since the Jacobian is $J(X \to Z) = |y|^{-1}$, the joint pdf of Y and Z is

$$\frac{x^{m-1}\exp(-x^2/2)}{2^{(m/2)-1}\Gamma(m/2)}\frac{\exp(-y^2/2)}{\sqrt{2\pi}}|y|^{-1}$$
$$=\frac{(z/y)^{m-1}\exp\{-(z/y)^2/2\}}{2^{(m/2)-1}\Gamma(m/2)}\frac{\exp(-y^2/2)}{\sqrt{2\pi}}|y|^{-1}$$
$$=\frac{(z/y)^{m-1}|y|^{-1}\exp[-\{(z/y)^2+y^2\}/2]}{2^{(m-1)/2}\pi^{1/2}\Gamma(m/2)},$$

where $z/y \geq 0$ by definition. The above result gives another expression of the pdf for Z

$$f_Z(z\,|\,m)=\frac{\int_{-\infty}^{\infty}(z/y)^{m-1}|y|^{-1}\exp[-\{(z/y)^2+y^2\}/2]\,\mathrm{d}y}{2^{(m-1)/2}\pi^{1/2}\Gamma(m/2)},$$

which is not simpler than that given earlier and can be shown to be equal to the previous one using $x = z/y$ and $J(Y \to X) = |\mathrm{d}y/\mathrm{d}x| = |z|x^{-2}$.

Remark S.2 The derivation of the pdf of Z suggests the corresponding pdf when X and Y are correlated. Let $Y = \rho X + (1-\rho^2)^{1/2}U$ $(\rho^2 \leq 1)$, where U is standard normally distributed and uncorrelated with X. Then, the correlation coefficient of X and Y becomes ρ. Consider the transformation from U to $Z = XY = X\{\rho X + (1-\rho^2)^{1/2}U\}$ with $J(U \to Z) = |x|^{-1}(1-\rho^2)^{-1/2}$. Since the joint pdf of X and Z is

$$(2\pi)^{-1}\exp\{-(x^2+u^2)/2\}J(U \to Z)$$
$$=(2\pi)^{-1}\exp[-\{x^2+(zx^{-1}-\rho x)^2(1-\rho^2)^{-1}\}/2]\,|x|^{-1}(1-\rho^2)^{-1/2}$$
$$=(2\pi)^{-1}(1-\rho^2)^{-1/2}\exp\left\{-\frac{x^2+z^2x^{-2}-2z\rho}{2(1-\rho^2)}\right\}|x|^{-1},$$

we have

$$f_Z(z)=(2\pi)^{-1}(1-\rho^2)^{-1/2}\int_{-\infty}^{\infty}\exp\left\{-\frac{x^2+z^2x^{-2}-2z\rho}{2(1-\rho^2)}\right\}|x|^{-1}\,\mathrm{d}x$$
$$=\pi^{-1}(1-\rho^2)^{-1/2}\int_0^{\infty}\exp\left\{-\frac{x^2+z^2x^{-2}-2z\rho}{2(1-\rho^2)}\right\}x^{-1}\mathrm{d}x,$$

which is seen as a special case of Pearson, Jeffery and Elderton (1929, Equation (iv)), Wishart and Bartlett (1932, Equation (12)), and Craig (1936, pp. 3-4) though these authors use the Bessel function of the second kind with imaginary argument (see McKay,1932; Watson, 1944/1995). It is found that when $\rho = 0$, the pdf becomes equal to that obtained earlier when $m = 1$. The mgf of Z is

$$
\begin{aligned}
\mathrm{M}_Z(t) &= \pi^{-1}(1-\rho^2)^{-1/2}\int_{-\infty}^{\infty}\int_0^{\infty}\exp\left\{-\frac{x^2+z^2x^{-2}-2z\rho}{2(1-\rho^2)}+zt\right\}x^{-1}\mathrm{d}x\mathrm{d}z \\
&= \pi^{-1}(1-\rho^2)^{-1/2}\int_0^{\infty}\int_{-\infty}^{\infty}\exp\left\{-\frac{z^2-2\{\rho+(1-\rho^2)t\}x^2z+x^4}{2(1-\rho^2)x^2}\right\}x^{-1}\mathrm{d}z\mathrm{d}x \\
&= \pi^{-1}(1-\rho^2)^{-1/2}\int_0^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{[z-\{\rho+(1-\rho^2)t\}x^2]^2}{2(1-\rho^2)x^2}\right)\mathrm{d}z \\
&\qquad\times\exp\left(-\frac{[1-\{\rho+(1-\rho^2)t\}^2]x^2}{2(1-\rho^2)}\right)x^{-1}\mathrm{d}x \\
&= (2/\pi)^{1/2}\int_0^{\infty}\exp\left(-\frac{[1-\{\rho+(1-\rho^2)t\}^2]x^2}{2(1-\rho^2)}\right)\mathrm{d}x \\
&= \frac{(1-\rho^2)^{1/2}}{[1-\{\rho+(1-\rho^2)t\}^2]^{1/2}}.
\end{aligned}
$$

The above result becomes $(1-t^2)^{-1/2}$ when $\rho=0$ as obtained earlier. An algebraically equal expression $\{1-2\rho t-(1-\rho^2)t^2\}^{-1/2}$ was given by Wishart and Bartlett (1932, Equation (9)), which supports the validity of $f_Z(z)$ given earlier.

Remark S.3 We deal with the sum of the products of correlated variables $\sum_{i=1}^m X_iY_i$, $(m\ge 2)$, where X_i and Y_i are standard normally distributed with $\mathrm{E}(X_iY_i)=\rho$ $(-1\le\rho\le 1)$ $(i=1,\dots,m)$ and independent of X_j and Y_j $(i\ne j)$. Redefine $X=\sum_{i=1}^m X_i^2$, $Y=\sum_{i=1}^m Y_i^2$ and $Z=\sum_{i=1}^m X_iY_i$ with the random matrix $\mathbf{S}^*=\begin{pmatrix} X & Z \\ Z & Y\end{pmatrix}$. Since $\mathbf{S}^*$ follows the Wishart distribution with the scale matrix $\boldsymbol{\Sigma}=\begin{pmatrix} 1 & \rho \\ \rho & 1\end{pmatrix}$ and *m* df, the density of $\mathbf{S}^*$ at $\mathbf{S}=\begin{pmatrix} x & z \\ z & y\end{pmatrix}$ with *p* = 2 becomes

$$
\begin{aligned}
w_p(\mathbf{S}\mid\boldsymbol{\Sigma},m) &= \frac{\exp\{-\mathrm{tr}(\boldsymbol{\Sigma}^{-1}\mathbf{S})/2\}\,|\,\mathbf{S}\,|^{(m-p-1)/2}}{2^{mp/2}\,|\,\boldsymbol{\Sigma}\,|^{m/2}\,\Gamma_p(m/2)} \\
&= \exp\left\{-\frac{x+y-2\rho z}{2(1-\rho^2)}\right\}\frac{(xy-z^2)^{(m-3)/2}}{2^m(1-\rho^2)^{m/2}\Gamma_2(m/2)}.
\end{aligned}
$$

Consider the variable transformation from Y to $U=XY-Z^2\ge 0$ with $J(Y\to U)=x^{-1}$. Then, the joint pdf of X, U and Z is

$$\exp\left\{-\frac{x+y-2\rho z}{2(1-\rho^2)}\right\}\frac{(xy-z^2)^{(m-3)/2}}{2^m(1-\rho^2)^{m/2}\Gamma_2(m/2)}J(Y\to U)$$
$$=\exp\left\{-\frac{x+(u+z^2)x^{-1}-2\rho z}{2(1-\rho^2)}\right\}\frac{u^{(m-3)/2}x^{-1}}{2^m(1-\rho^2)^{m/2}\Gamma_2(m/2)}$$
$$=\exp\left\{-\frac{u}{2(1-\rho^2)x}\right\}\frac{u^{(m-3)/2}x^{-1}}{2^m(1-\rho^2)^{m/2}\Gamma_2(m/2)}\exp\left\{-\frac{x+z^2x^{-1}-2\rho z}{2(1-\rho^2)}\right\},$$
which gives the marginal density of X and Z as
$$\int_0^\infty \exp\left\{-\frac{u}{2(1-\rho^2)x}\right\}\frac{u^{(m-3)/2}x^{-1}}{2^m(1-\rho^2)^{m/2}\Gamma_2(m/2)}\mathrm{d}u\ \exp\left\{-\frac{x+z^2x^{-1}-2\rho z}{2(1-\rho^2)}\right\}$$
$$=\frac{\{2(1-\rho^2)x\}^{(m-1)/2}\Gamma\{(m-1)/2\}x^{-1}}{2^m(1-\rho^2)^{m/2}\Gamma_2(m/2)}\exp\left\{-\frac{x+z^2x^{-1}-2\rho z}{2(1-\rho^2)}\right\}$$
$$=\frac{x^{(m-3)/2}}{2^{(m+1)/2}(1-\rho^2)^{1/2}\pi^{1/2}\Gamma(m/2)}\exp\left\{-\frac{x+z^2x^{-1}-2\rho z}{2(1-\rho^2)}\right\}.$$
From this result, the pdf of Z is derived as
$$f_Z(z\,|\,m)=\int_0^\infty \frac{x^{(m-3)/2}}{2^{(m+1)/2}(1-\rho^2)^{1/2}\pi^{1/2}\Gamma(m/2)}\exp\left\{-\frac{x+z^2x^{-1}-2\rho z}{2(1-\rho^2)}\right\}\mathrm{d}x\,.$$
Let $X=V^2$ with $J(X\to V)=2v$. Then, the above result becomes
$$f_Z(z\,|\,m)=\int_0^\infty \frac{x^{(m-3)/2}}{2^{(m+1)/2}(1-\rho^2)^{1/2}\pi^{1/2}\Gamma(m/2)}\exp\left\{-\frac{x+z^2x^{-1}-2\rho z}{2(1-\rho^2)}\right\}2v\mathrm{d}v$$
$$=\int_0^\infty \frac{v^{m-2}}{2^{(m-1)/2}(1-\rho^2)^{1/2}\pi^{1/2}\Gamma(m/2)}\exp\left\{-\frac{v^2+z^2v^{-2}-2\rho z}{2(1-\rho^2)}\right\}\mathrm{d}v \quad .$$
$$=\int_0^\infty \frac{x^{m-2}}{2^{(m-1)/2}(1-\rho^2)^{1/2}\pi^{1/2}\Gamma(m/2)}\exp\left\{-\frac{x^2+z^2x^{-2}-2\rho z}{2(1-\rho^2)}\right\}\mathrm{d}x,$$
where the last result is given by the redefinition of $X=V$. Note that in the above density $m\ge 2$ is assumed. Though $\Gamma_2(m/2)$ when $m=1$ is not defined, it is found that the derived density when $m=1$ becomes
$$f_Z(z\,|\,m=1)=\frac{1}{\pi(1-\rho^2)^{1/2}}\int_0^\infty \exp\left\{-\frac{x^2+z^2x^{-2}-2\rho z}{2(1-\rho^2)}\right\}x^{-1}\mathrm{d}x$$
as obtained earlier for the product of the correlated standard normal variables.

Remark S.4 The mgf of the sum of the products of correlated variables $Z=\sum_{i=1}^m X_iY_i$ $(m\ge 2)$ as defined in Remark S.3 is obtained. Using the pdf of Z, we have

$$
\begin{aligned}
&\mathrm{M}_Z(t\,|\,m)\\
&=\int_{-\infty}^{\infty}\int_0^{\infty}\frac{x^{m-2}}{2^{(m-1)/2}(1-\rho^2)^{1/2}\pi^{1/2}\Gamma(m/2)}\exp\left\{-\frac{x^2+z^2x^{-2}-2\rho z}{2(1-\rho^2)}\right\}\exp(tz)\,\mathrm{d}x\,\mathrm{d}z\\
&=\int_0^{\infty}\frac{x^{m-2}}{2^{(m-1)/2}(1-\rho^2)^{1/2}\pi^{1/2}\Gamma(m/2)}\int_{-\infty}^{\infty}\exp\left\{-\frac{x^2+z^2x^{-2}-2\rho z-2(1-\rho^2)tz}{2(1-\rho^2)}\right\}\mathrm{d}z\,\mathrm{d}x\\
&=\int_0^{\infty}\frac{x^{m-2}}{2^{(m-1)/2}(1-\rho^2)^{1/2}\pi^{1/2}\Gamma(m/2)}\\
&\quad\times\int_{-\infty}^{\infty}\exp\left(-\frac{[z-\{\rho+(1-\rho^2)t\}x^2]^2+x^4-\{\rho+(1-\rho^2)t\}^2x^4}{2(1-\rho^2)x^2}\right)\mathrm{d}z\,\mathrm{d}x\\
&=\int_0^{\infty}\frac{x^{m-2}}{2^{(m-1)/2}(1-\rho^2)^{1/2}\pi^{1/2}\Gamma(m/2)}(2\pi)^{1/2}(1-\rho^2)^{1/2}x\\
&\qquad\times\exp\left(-\frac{[1-\{\rho+(1-\rho^2)t\}^2]x^4}{2(1-\rho^2)x^2}\right)\mathrm{d}x\\
&=\int_0^{\infty}\frac{x^{m-1}}{2^{(m-2)/2}\Gamma(m/2)}\exp\left(-\frac{[1-\{\rho+(1-\rho^2)t\}^2]x^2}{2(1-\rho^2)}\right)\mathrm{d}x\\
&=\int_0^{\infty}\frac{v^{(m-1)/2}}{2^{(m-2)/2}\Gamma(m/2)}\exp\left(-\frac{[1-\{\rho+(1-\rho^2)t\}^2]v}{2(1-\rho^2)}\right)\frac{v^{-1/2}}{2}\mathrm{d}v\\
&=\int_0^{\infty}\frac{v^{(m-2)/2}}{2^{m/2}\Gamma(m/2)}\exp\left(-\frac{[1-\{\rho+(1-\rho^2)t\}^2]v}{2(1-\rho^2)}\right)\mathrm{d}v\\
&=\frac{(1-\rho^2)^{m/2}}{[1-\{\rho+(1-\rho^2)t\}^2]^{m/2}},
\end{aligned}
$$

which is expected since $Z=\sum_{i=1}^{m}X_iY_i$ is the sum of m independent identically distributed terms, where the mgf of each term was obtained as

$$\frac{(1-\rho^2)^{1/2}}{[1-\{\rho+(1-\rho^2)t\}^2]^{1/2}}.$$

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