

Expository supplement I to the paper “Asymptotic expansions for the estimators of Lagrange multipliers and associated parameters by the maximum likelihood and weighted score methods”

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This article gives the first half of an expository supplement to Ogasawara (2015).

1. Asymptotic expansions of the estimators with restrictions for model identification

In this section, the estimator $\hat{\boldsymbol{\theta}}_W$ with restrictions for model identification is dealt with. Recall that in this case $\hat{\boldsymbol{\eta}}_W = \mathbf{0}$. Setting $\hat{\boldsymbol{\eta}}_W = \mathbf{0}$ in (A1.4), we have

$$\begin{aligned}
 \mathbf{0} &= \begin{pmatrix} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \\ \mathbf{0} \end{pmatrix}_{O_p(n^{-1/2})} + \begin{pmatrix} n^{-1} \mathbf{q}_0^* \\ \mathbf{0} \end{pmatrix}_{O(n^{-1})} + \left\{ \boldsymbol{\Lambda}_0^* \begin{pmatrix} \hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0 \\ \mathbf{0} \end{pmatrix} \right\}_{O_p(n^{-1/2})} \\
 &+ \left\{ \begin{pmatrix} \mathbf{L}_0 - \boldsymbol{\Lambda}_0 \\ \mathbf{0} \end{pmatrix} (\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0) \right\}_{O_p(n^{-1})} + \left\{ n^{-1} \frac{\partial \mathbf{q}_0^*}{\partial \boldsymbol{\theta}_0'} (\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0) \right\}_{O_p(n^{-3/2})} \quad (\text{S1.1}) \\
 &+ \left\{ \frac{1}{2} \begin{pmatrix} \frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<2>}} \\ \frac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0')^{<2>}} \end{pmatrix} (\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0)^{<2>} \right\}_{O_p(n^{-1})}
 \end{aligned}$$

$$+ \left\{ \frac{1}{6} \left(\begin{array}{c} \frac{\partial^4 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<3>}} \\ \frac{\partial^3 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0')^{<3>}} \end{array} \right) (\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0)^{<3>} \right\}_{O_p(n^{-3/2})} + O_p(n^{-2}).$$

From the above result,

$$\begin{aligned} \left(\begin{array}{c} \hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0 \\ \mathbf{0} \end{array} \right) &= - \left\{ \boldsymbol{\Lambda}_0^{*-1} \left(\begin{array}{c} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \\ \mathbf{0} \end{array} \right) \right\}_{O_p(n^{-1/2})} - \left\{ n^{-1} \boldsymbol{\Lambda}_0^{*-1} \left(\begin{array}{c} \mathbf{q}_0^* \\ \mathbf{0} \end{array} \right) \right\}_{O(n^{-1})} \\ &- \left\{ \boldsymbol{\Lambda}_0^{*-1} \left(\begin{array}{c} \mathbf{L}_0 - \boldsymbol{\Lambda}_0 \\ \mathbf{0} \end{array} \right) (\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0) \right\}_{O_p(n^{-1})} \\ &- \left\{ n^{-1} \boldsymbol{\Lambda}_0^{*-1} \left(\begin{array}{c} \frac{\partial \mathbf{q}_0^*}{\partial \boldsymbol{\theta}_0'} (\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0) \\ \mathbf{0} \end{array} \right) \right\}_{O_p(n^{-3/2})} \\ &- \left\{ \frac{1}{2} \boldsymbol{\Lambda}_0^{*-1} \left(\begin{array}{c} \mathbf{E}_T(\mathbf{J}_0^{(3)}) \\ \frac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0')^{<2>}} \end{array} \right) (\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0)^{<2>} \right\}_{O_p(n^{-1})} \\ &- \left\{ \frac{1}{2} \boldsymbol{\Lambda}_0^{*-1} \left(\begin{array}{c} \mathbf{J}_0^{(3)} - \mathbf{E}_T(\mathbf{J}_0^{(3)}) \\ \mathbf{0} \end{array} \right) (\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0)^{<2>} \right\}_{O_p(n^{-3/2})} \\ &- \left\{ \frac{1}{6} \boldsymbol{\Lambda}_0^{*-1} \left(\begin{array}{c} \mathbf{E}_T(\mathbf{J}_0^{(4)}) \\ \frac{\partial^3 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0')^{<3>}} \end{array} \right) (\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0)^{<3>} \right\}_{O_p(n^{-3/2})} + O_p(n^{-2}). \end{aligned} \tag{S1.2}$$

Recalling $\mathbf{M} \equiv \mathbf{L}_0 - \boldsymbol{\Lambda}_0$, Equation (S1.2) gives the following result:

$$\begin{aligned}
\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0 &= - \left(\boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)_{O_p(n^{-1/2})} - (n^{-1} \boldsymbol{\Lambda}_0^{(11)} \mathbf{q}_0^*)_{O(n^{-1})} \\
&+ \left\{ \boldsymbol{\Lambda}_0^{(11)} \mathbf{M} \boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right\}_{O_p(n^{-1})} \\
&- \left\{ \frac{1}{2} (\boldsymbol{\Lambda}_0^{(11)} \quad \boldsymbol{\Lambda}_0^{(12)}) \begin{pmatrix} \mathbf{E}_T(\mathbf{J}_0^{(3)}) \\ \frac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0')^{<2>}} \end{pmatrix} \left(\boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>} \right\}_{O_p(n^{-1})} \\
&+ (n^{-1} \boldsymbol{\Lambda}_0^{(11)} \mathbf{M} \boldsymbol{\Lambda}_0^{(11)} \mathbf{q}_0^*)_{O_p(n^{-3/2})} - \left(\boldsymbol{\Lambda}_0^{(11)} \mathbf{M} \boldsymbol{\Lambda}_0^{(11)} \mathbf{M} \boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)_{O_p(n^{-3/2})} \\
&+ \left\{ \frac{1}{2} \boldsymbol{\Lambda}_0^{(11)} \mathbf{M} (\boldsymbol{\Lambda}_0^{(11)} \quad \boldsymbol{\Lambda}_0^{(12)}) \begin{pmatrix} \mathbf{E}_T(\mathbf{J}_0^{(3)}) \\ \frac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0')^{<2>}} \end{pmatrix} \left(\boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>} \right\}_{O_p(n^{-3/2})} \\
&+ \left(n^{-1} \boldsymbol{\Lambda}_0^{(11)} \frac{\partial \mathbf{q}_0^*}{\partial \boldsymbol{\theta}_0'} \boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)_{O_p(n^{-3/2})} \\
&- \left[\begin{matrix} (\boldsymbol{\Lambda}_0^{(11)} \boldsymbol{\Lambda}_0^{(12)}) \begin{pmatrix} \mathbf{E}_T(\mathbf{J}_0^{(3)}) \\ \frac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0')^{<2>}} \end{pmatrix} \\ (\text{A}) \end{matrix} \right] \left[\begin{matrix} \left(\boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right) \\ (\text{B}) \end{matrix} \right] \\
&\otimes \left\{ n^{-1} \boldsymbol{\Lambda}_0^{(11)} \mathbf{q}_0^* - \boldsymbol{\Lambda}_0^{(11)} \mathbf{M} \boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right. \\
&\quad \left. + \frac{1}{2} (\boldsymbol{\Lambda}_0^{(11)} \boldsymbol{\Lambda}_0^{(12)}) \begin{pmatrix} \mathbf{E}_T(\mathbf{J}_0^{(3)}) \\ \frac{\partial^2 \mathbf{h}_0}{(\partial \boldsymbol{\theta}_0')^{<2>}} \end{pmatrix} \left(\boldsymbol{\Lambda}_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>} \right\} \left[\begin{matrix} \\ (\text{B}) (\text{A}) \end{matrix} \right]_{O_p(n^{-3/2})} \tag{S1.3}
\end{aligned}$$

$$\begin{aligned}
& - \left\{ \frac{1}{2} \Lambda_0^{(11)} (\mathbf{J}_0^{(3)} - \mathbb{E}_T(\mathbf{J}_0^{(3)})) \left(\Lambda_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{\langle 2 \rangle} \right\}_{O_p(n^{-3/2})} \\
& + \left\{ \frac{1}{6} (\Lambda_0^{(11)} \ \Lambda_0^{(12)}) \left(\frac{\mathbb{E}_T(\mathbf{J}_0^{(4)})}{(\partial \boldsymbol{\theta}_0')^{\langle 3 \rangle}} \right) \left(\Lambda_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{\langle 3 \rangle} \right\}_{O_p(n^{-3/2})} + O_p(n^{-2}) \\
& \equiv \sum_{i=1}^3 \Lambda_W^{(i)} \mathbf{I}_0^{(i)} - n^{-1} \Lambda_0^{(11)} \mathbf{q}_0^* + O_p(n^{-2}) \\
& (\Lambda_W^{(i)} = O(1), \mathbf{I}_0^{(i)} = O_p(n^{-i/2}), i = 1, 2, 3).
\end{aligned}$$

In (S1.3),

$$\Lambda_W^{(1)} \mathbf{I}_0^{(1)} = \Lambda_{ML}^{(1)} \mathbf{I}_0^{(1)} = -\Lambda_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \quad (\text{S1.4})$$

where $\Lambda_W^{(1)} = \Lambda_{ML}^{(1)} = -\Lambda_0^{(11)}$, $\mathbf{I}_0^{(1)} = \partial \bar{l} / \partial \boldsymbol{\theta}_0$,

$$\begin{aligned}
& \Lambda_W^{(2)} \mathbf{I}_0^{(2)} = \Lambda_{ML}^{(2)} \mathbf{I}_0^{(2)} \\
& = \Lambda_0^{(11)} \mathbf{M} \Lambda_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} - \frac{1}{2} (\Lambda_0^{(11)} \ \Lambda_0^{(12)}) \left(\frac{\mathbb{E}_T(\mathbf{J}_0^{(3)})}{(\partial \boldsymbol{\theta}_0')^{\langle 2 \rangle}} \right) \left(\Lambda_0^{(11)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{\langle 2 \rangle} \\
& \quad (\text{S1.5})
\end{aligned}$$

$$= (\Lambda_{ML}^{(2-1)} \ \Lambda_{ML}^{(2-2)}) \left\{ \mathbf{v}(\mathbf{M})' \otimes \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'}, \left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right)^{\langle 2 \rangle} \right\}',$$

where $\Lambda_{ML}^{(2)} = (\Lambda_{ML}^{(2-1)} \ \Lambda_{ML}^{(2-2)})$ with $\Lambda_{ML}^{(2-1)}$ and $\Lambda_{ML}^{(2-2)}$ being $q \times \{q^2(q+1)/2\}$ and $q \times q^2$ submatrices, respectively,

$$(\Lambda_{\text{ML}}^{(2-1)})_{\cdot(ijk^*)} = \sum_{(ij)}^2 (\Lambda_0^{(11)})_{\cdot i} \frac{2-\delta_{ij}}{2} (\Lambda_0^{(11)})_{jk^*}$$

$$(1 \leq i \leq j \leq q; k^* = 1, \dots, q),$$

$$(\Lambda_{\text{ML}}^{(2-2)})_{\cdot(ij)} = -\frac{1}{2} (\Lambda_0^{(11)} \Lambda_0^{(12)}) \left(\frac{\mathbf{E}_T(\mathbf{J}_0^{(3)})}{(\partial \boldsymbol{\theta}_0')^{\langle 2 \rangle}} \right) \{ (\Lambda_0^{(11)})_{\cdot i} \otimes (\Lambda_0^{(11)})_{\cdot j} \}$$

$$(i, j = 1, \dots, q),$$

$$\Lambda_{\text{W}}^{(3)} \mathbf{I}_0^{(3)} = \Lambda_{\text{ML}}^{(3)} \mathbf{I}_0^{(3)} + \Lambda_{\text{W}}^{(\Delta)} n^{-1} \mathbf{I}_0^{(\text{W})}, \quad (\text{S1.6})$$

$$\Lambda_{\text{ML}}^{(3)} \mathbf{I}_0^{(3)} = (\Lambda_{\text{ML}}^{(3-1)} \Lambda_{\text{ML}}^{(3-2)} \Lambda_{\text{ML}}^{(3-3)} \Lambda_{\text{ML}}^{(3-4)})$$

$$\times \left\{ \mathbf{v}(\mathbf{M})'^{\langle 2 \rangle} \otimes \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'}, \mathbf{v}(\mathbf{M})' \otimes \left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right)^{\langle 2 \rangle} \right\},$$

$$\text{vec} \{ (\mathbf{J}_0^{(3)} - \mathbf{E}_T(\mathbf{J}_0^{(3)})) \}' \otimes \left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right)^{\langle 2 \rangle}, \left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right)^{\langle 3 \rangle} \right\}',$$

$$\Lambda_{\text{W}}^{(\Delta)} n^{-1} \mathbf{I}_0^{(\text{W})} = (\Lambda_{\text{W}}^{(\Delta-1)} \Lambda_{\text{W}}^{(\Delta-2)}) n^{-1} \left\{ \mathbf{v}(\mathbf{M})', \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right\}',$$

$$(\Lambda_{\text{ML}}^{(3-1)})_{\cdot(ijk^*l^*m)} = -\sum_{(ij)}^2 \sum_{(k^*l^*)}^2 (\Lambda_0^{(11)})_{\cdot i} \frac{2-\delta_{ij}}{2} (\Lambda_0^{(11)})_{jk^*} \frac{2-\delta_{k^*l^*}}{2} (\Lambda_0^{(11)})_{l^*m}$$

$$(1 \leq i \leq j \leq q; 1 \leq k^* \leq l^* \leq q; m = 1, \dots, q),$$

$$\begin{aligned}
(\Lambda_{\text{ML}}^{(3-2)})_{\cdot(ijk^*l^*)} &= \sum_{(ij)}^2 \frac{1}{2} (\Lambda_0^{(11)})_{\cdot i} \frac{2 - \delta_{ij}}{2} \\
&\times \left\{ (\Lambda_0^{(11)} \Lambda_0^{(12)}) \left(\frac{\mathbf{E}_T(\mathbf{J}_0^{(3)})}{(\partial \boldsymbol{\theta}_0')^{<2>}} \right) \{ (\Lambda_0^{(11)})_{\cdot k^*} \otimes (\Lambda_0^{(11)})_{\cdot l^*} \} \right\}_j \\
&+ (\Lambda_0^{(11)} \Lambda_0^{(12)}) \left(\frac{\mathbf{E}_T(\mathbf{J}_0^{(3)})}{(\partial \boldsymbol{\theta}_0')^{<2>}} \right) \left[(\Lambda_0^{(11)})_{\cdot k^*} \otimes \sum_{(ij)}^2 \left\{ (\Lambda_0^{(11)})_{\cdot i} \frac{2 - \delta_{ij}}{2} (\Lambda_0^{(11)})_{jl^*} \right\} \right]
\end{aligned}$$

$$(1 \leq i \leq j \leq q; k^*, l^* = 1, \dots, q),$$

$$(\Lambda_{\text{ML}}^{(3-3)})_{\cdot(ijk^*l^*m)} = -\frac{1}{2} (\Lambda_0^{(11)})_{\cdot i} (\Lambda_0^{(11)})_{jl^*} (\Lambda_0^{(11)})_{k^*m}$$

$$(i, j, k^*, l^*, m = 1, \dots, q),$$

$$\begin{aligned}
(\Lambda_{\text{ML}}^{(3-4)})_{\cdot(ijk^*)} &= -\frac{1}{2} (\Lambda_0^{(11)} \Lambda_0^{(12)}) \left(\frac{\mathbf{E}_T(\mathbf{J}_0^{(3)})}{(\partial \boldsymbol{\theta}_0')^{<2>}} \right)_{(A)} \left[(\Lambda_0^{(11)})_{\cdot i} \right. \\
&\otimes \left. \left\{ (\Lambda_0^{(11)} \Lambda_0^{(12)}) \left(\frac{\mathbf{E}_T(\mathbf{J}_0^{(3)})}{(\partial \boldsymbol{\theta}_0')^{<2>}} \right) \{ (\Lambda_0^{(11)})_{\cdot j} \otimes (\Lambda_0^{(11)})_{\cdot k} \} \right\}_{(B)} \right]_{(A)} \\
&+ \frac{1}{6} (\Lambda_0^{(11)} \Lambda_0^{(12)}) \left(\frac{\mathbf{E}_T(\mathbf{J}_0^{(4)})}{(\partial \boldsymbol{\theta}_0')^{<3>}} \right) \{ (\Lambda_0^{(11)})_{\cdot i} \otimes (\Lambda_0^{(11)})_{\cdot j} \otimes (\Lambda_0^{(11)})_{\cdot k^*} \}
\end{aligned}$$

$$(i, j, k^* = 1, \dots, q),$$

$$(\Lambda_W^{(\Delta-1)})_{\cdot(ij)} = \sum_{(ij)}^2 (\Lambda_0^{(11)})_{\cdot i} \frac{2 - \delta_{ij}}{2} (\Lambda_0^{(11)} \mathbf{q}_0^*)_j$$

$$(1 \leq i \leq j \leq q),$$

$$(\Lambda_W^{(\Delta-2)})_{\cdot i} = \Lambda_0^{(11)} \frac{\partial \mathbf{q}_0^*}{\partial \boldsymbol{\theta}_0'} (\Lambda_0^{(11)})_{\cdot i}$$

$$- (\Lambda_0^{(11)} \Lambda_0^{(12)}) \left(\frac{\mathbf{E}_T(\mathbf{J}_0^{(3)})}{(\partial \boldsymbol{\theta}_0')^{\langle 2 \rangle}} \right) \{ (\Lambda_0^{(11)})_{\cdot i} \otimes (\Lambda_0^{(11)} \mathbf{q}_0^*) \}$$

$$(i = 1, \dots, q).$$

Equation (S1.3) is alternatively written as

$$\begin{aligned} \hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0 &= \sum_{i=1}^3 \Lambda_{ML}^{(i)} \mathbf{I}_0^{(i)} + \Lambda_W^{(\Delta)} n^{-1} \mathbf{I}_0^{(w)} - n^{-1} \Lambda_0^{(11)} \mathbf{q}_0^* + O_p(n^{-2}) \\ &= \hat{\boldsymbol{\theta}}_{ML} - \boldsymbol{\theta}_0 + \Lambda_W^{(\Delta)} n^{-1} \mathbf{I}_0^{(w)} - n^{-1} \Lambda_0^{(11)} \mathbf{q}_0^* + O_p(n^{-2}). \end{aligned} \quad (\text{S1.7})$$

2. Properties of augmented matrices

Define column-wise q blocks of a matrix \mathbf{A} as $\mathbf{A}_{(\cdot j)}$ ($j = 1, \dots, q$) i.e., $\mathbf{A} = (\mathbf{A}_{(\cdot 1)} \mathbf{A}_{(\cdot 2)} \cdots \mathbf{A}_{(\cdot q)})$. Similarly, define row-wise p blocks of \mathbf{A} as $\mathbf{A}_{(i \cdot)}$ ($i = 1, \dots, p$) i.e., $\mathbf{A} = (\mathbf{A}_{(1 \cdot)}' \mathbf{A}_{(2 \cdot)}' \cdots \mathbf{A}_{(p \cdot)}')'$. Let $\mathbf{B} = (\mathbf{B}_{(\cdot 1)} \mathbf{B}_{(\cdot 2)})$, where $\mathbf{B}_{(\cdot 1)}$ and $\mathbf{B}_{(\cdot 2)}$ are $b \times q$ and $b \times r$ submatrices, respectively;

$q > r$, $q = r$ or $q < r$. Post-multiply \mathbf{B} by $\mathbf{G} = \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ \mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix}$. Then, \mathbf{BG}

becomes $(\mathbf{B}_{(\cdot 1)} + \mathbf{B}_{(\cdot 2)} \mathbf{C}' : \mathbf{B}_{(\cdot 2)})$ with the colon being used to show separation for clarity. Matrix \mathbf{B} is restored from \mathbf{BG} by subtracting $\mathbf{B}_{(\cdot 2)} \mathbf{C}'$ from the first

column-wise block of \mathbf{BG} , which is given by $\mathbf{G}^* = \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ -\mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix}$. Since

$\mathbf{BGG}^* = \mathbf{B}$, $\mathbf{G}^* = \mathbf{G}^{-1}$. That is, we have elementary results:

Lemma S1. $\begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ \mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ -\mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix}$ and

$$\begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{C} \\ \mathbf{O} & \mathbf{I}_{(r)} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{I}_{(q)} & -\mathbf{C} \\ \mathbf{O} & \mathbf{I}_{(r)} \end{pmatrix}.$$

Let $\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}$ be a non-singular $(q+r) \times (q+r)$ matrix, where the $q \times q$ diagonal block \mathbf{B} may be asymmetric and singular.

Lemma S2. *The first column-wise and row-wise blocks of $\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$ are*

equal to those of $\begin{pmatrix} \mathbf{B} + \mathbf{DC}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$, respectively. Equivalently, the two matrices are different only in their second diagonal blocks.

Proof. Since $\left\{ \begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ \mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix} \right\}^{-1} = \begin{pmatrix} \mathbf{B} + \mathbf{DC}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$,

$$\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ \mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix} \begin{pmatrix} \mathbf{B} + \mathbf{DC}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}. \text{ Using Lemma S1,}$$

$$\begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ -\mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix} \begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{B} + \mathbf{DC}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}. \text{ Noting that pre-multiplication of a}$$

matrix by $\begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ -\mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix}$ does not change the first row-wise block of the

multiplied matrix, we find that the first row-wise blocks of the inverted matrices are equal. Similarly, from

$$\left\{ \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{D} \\ \mathbf{O} & \mathbf{I}_{(r)} \end{pmatrix} \begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix} \right\}^{-1} = \begin{pmatrix} \mathbf{B} + \mathbf{DC}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} \text{ we have}$$

$$\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{B} + \mathbf{DC}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{D} \\ \mathbf{O} & \mathbf{I}_{(r)} \end{pmatrix}. \text{ Using Lemma S1,}$$

$\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{I}_{(q)} & -\mathbf{D} \\ \mathbf{O} & \mathbf{I}_{(r)} \end{pmatrix} = \begin{pmatrix} \mathbf{B} + \mathbf{DC}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$, which shows that the first column-wise blocks of the inverted matrices are equal. Q.E.D.

When \mathbf{B} is non-singular and when $q \geq r$, $\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$ is straightforwardly obtained by the formula of the inverse of a partitioned square matrix

$\begin{pmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{E}^{-1} + \mathbf{E}^{-1}\mathbf{F}\mathbf{J}\mathbf{G}\mathbf{E}^{-1} & -\mathbf{E}^{-1}\mathbf{F}\mathbf{J} \\ -\mathbf{J}\mathbf{G}\mathbf{E}^{-1} & \mathbf{J} \end{pmatrix}$, where $\mathbf{J} = (\mathbf{H} - \mathbf{G}\mathbf{E}^{-1}\mathbf{F})^{-1}$ with the assumption of the existence of the inverses, which gives

$$\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{B}^{-1} - \mathbf{B}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{B}^{-1} & \mathbf{B}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1} \\ (\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{B}^{-1} & -(\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1} \end{pmatrix}.$$

When \mathbf{B} is possibly singular, and $\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}$ with $q \geq r$ and $\mathbf{B} + \mathbf{DC}'$ are non-singular, we have

Lemma S3. *Let $\mathbf{A} = \mathbf{B} + \mathbf{DC}'$. Then,*

$$\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{A}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1} \\ (\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{I}_{(r)} - (\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1} \end{pmatrix}.$$

Proof. Using the result in the proof of Lemma S2, and the formula of the inverse of a partitioned matrix,

$$\begin{aligned} \begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} &= \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ \mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix} \begin{pmatrix} \mathbf{B} + \mathbf{DC}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} \\ \mathbf{C}' & \mathbf{I}_{(r)} \end{pmatrix} \begin{pmatrix} \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{A}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1} \\ (\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}^{-1} & -(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{A}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1} \\ (\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{I}_{(r)} - (\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1} \end{pmatrix}. \quad \text{Q.E.D.} \end{aligned}$$

In the case of the augmented information matrix per observation \mathbf{I}_0^* with $q > r$, we have

Theorem S1. Let $\mathbf{I}_0^* = \begin{pmatrix} \mathbf{I}_0 & -\mathbf{H}_0 \\ -\mathbf{H}_0' & \mathbf{O} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{I}_0 & \mathbf{C} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{I}_0^{(11)} & \mathbf{I}_0^{(12)} \\ \mathbf{I}_0^{(21)} & \mathbf{I}_0^{(22)} \end{pmatrix}^{-1}$, where

\mathbf{I}_0 may be singular; $\mathbf{A} = \mathbf{I}_0 + \mathbf{C}\mathbf{C}'$ and $\mathbf{C}'\mathbf{A}^{-1}\mathbf{C}$ are non-singular. Then,

$$\mathbf{I}_0^{*-1} = \begin{pmatrix} \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1} \\ (\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{I}_{(r)} - (\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1} \end{pmatrix}.$$

Proof. Use Lemma S3 with $\mathbf{B} = \mathbf{I}_0$ and $\mathbf{D} = \mathbf{C}$. Q.E.D.

It is known that the asymptotic covariance matrix of $(n^{1/2}\hat{\boldsymbol{\theta}}_w', n^{-1/2}\hat{\boldsymbol{\eta}}_w')$

under correct model specification is given by $\mathbf{I}_0^{*-1} \begin{pmatrix} \mathbf{I}_0 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \mathbf{I}_0^{*-1}$, whose

alternative expression (see Silvey, 1959, Lemma 6; Jennrich, 1974; Lee, 1979) is given by the following:

Corollary S1. When \mathbf{I}_0 in the augmented matrix may be singular,

$$\mathbf{I}_0^{*-1} \begin{pmatrix} \mathbf{I}_0 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \mathbf{I}_0^{*-1} = \begin{pmatrix} \mathbf{I}_0^{(11)} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I}_0^{(22)} \end{pmatrix}.$$

Proof.

$$\begin{aligned} \mathbf{I}_0^{*-1} \begin{pmatrix} \mathbf{I}_0 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \mathbf{I}_0^{*-1} &= \begin{pmatrix} \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1} \\ (\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1} \end{pmatrix} (\mathbf{A} - \mathbf{C}\mathbf{C}') \\ &\times \begin{pmatrix} \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{O} \\ \mathbf{O} & (\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1} - \mathbf{I}_{(r)} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{I}_0^{(11)} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I}_0^{(22)} \end{pmatrix} \text{ Q.E.D.} \end{aligned}$$

When \mathbf{I}_0 is non-singular, since \mathbf{I}_0^{*-1} is obtained by the formula of the inverse of a partitioned matrix as

$$\mathbf{I}_0^{*-1} = \begin{pmatrix} \mathbf{I}_0 & \mathbf{C} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{I}_0^{-1} - \mathbf{I}_0^{-1}\mathbf{C}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{I}_0^{-1} & \mathbf{I}_0^{-1}\mathbf{C}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1} \\ (\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{I}_0^{-1} & -(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1} \end{pmatrix}, \text{ it}$$

follows that

$$\begin{aligned} \mathbf{I}_0^{*-1} \begin{pmatrix} \mathbf{I}_0 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \mathbf{I}_0^{*-1} &= \begin{pmatrix} \mathbf{I}_0^{-1} - \mathbf{I}_0^{-1} \mathbf{C} (\mathbf{C}' \mathbf{I}_0^{-1} \mathbf{C})^{-1} \mathbf{C}' \mathbf{I}_0^{-1} & \mathbf{O} \\ \mathbf{O} & (\mathbf{C}' \mathbf{I}_0^{-1} \mathbf{C})^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{I}_0^{(11)} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I}_0^{(22)} \end{pmatrix}, \end{aligned}$$

which is also known (Aitchison & Silvey, 1958, Theorem 2).

The result $\mathbf{I}_0^{(11)} \mathbf{I}_0 \mathbf{I}_0^{(11)} = \mathbf{I}_0^{(11)}$ in Corollary S1 shows that \mathbf{I}_0 is a generalized inverse of $\mathbf{I}_0^{(11)}$. However, the reverse is not true (Jennrich, 1974), which is shown as follows.

$$\begin{aligned} \mathbf{I}_0 \mathbf{I}_0^{(11)} \mathbf{I}_0 &= (\mathbf{A} - \mathbf{C} \mathbf{C}') \{ \mathbf{A}^{-1} - \mathbf{A}^{-1} \mathbf{C} (\mathbf{C}' \mathbf{A}^{-1} \mathbf{C})^{-1} \mathbf{C}' \mathbf{A}^{-1} \} (\mathbf{A} - \mathbf{C} \mathbf{C}') \\ &= \mathbf{A} - 2\mathbf{C} \mathbf{C}' + \mathbf{C} \mathbf{C}' \mathbf{A}^{-1} \mathbf{C} \mathbf{C}' \\ &\quad - (\mathbf{A} - \mathbf{C} \mathbf{C}') \mathbf{A}^{-1} \mathbf{C} (\mathbf{C}' \mathbf{A}^{-1} \mathbf{C})^{-1} \mathbf{C}' \mathbf{A}^{-1} (\mathbf{A} - \mathbf{C} \mathbf{C}') \\ &= \mathbf{A} - 2\mathbf{C} \mathbf{C}' + \mathbf{C} \mathbf{C}' \mathbf{A}^{-1} \mathbf{C} \mathbf{C}' \\ &\quad - \{ \mathbf{C} (\mathbf{C}' \mathbf{A}^{-1} \mathbf{C})^{-1} \mathbf{C}' - 2\mathbf{C} \mathbf{C}' + \mathbf{C} \mathbf{C}' \mathbf{A}^{-1} \mathbf{C} \mathbf{C}' \} \\ &= \mathbf{A} - \mathbf{C} (\mathbf{C}' \mathbf{A}^{-1} \mathbf{C})^{-1} \mathbf{C}' \\ &= \mathbf{I}_0 + \mathbf{C} [\mathbf{I}_{(r)} - \{ \mathbf{C}' (\mathbf{I}_0 + \mathbf{C} \mathbf{C}')^{-1} \mathbf{C} \}^{-1}] \mathbf{C}' \\ &< \mathbf{I}_0, \end{aligned}$$

which indicates that $\mathbf{I}_0^{(11)}$ is not a generalized inverse of \mathbf{I}_0 .

Noting that Theorem S1 holds when \mathbf{I}_0 is non-singular as well as when \mathbf{I}_0 is singular, we have

Corollary S2. *When \mathbf{I}_0 is non-singular and $\mathbf{A} = \mathbf{I}_0 + \mathbf{C} \mathbf{C}'$,*

$$\begin{aligned} &\begin{pmatrix} \mathbf{A}^{-1} - \mathbf{A}^{-1} \mathbf{C} (\mathbf{C}' \mathbf{A}^{-1} \mathbf{C})^{-1} \mathbf{C}' \mathbf{A}^{-1} & \mathbf{A}^{-1} \mathbf{C} (\mathbf{C}' \mathbf{A}^{-1} \mathbf{C})^{-1} \\ (\mathbf{C}' \mathbf{A}^{-1} \mathbf{C})^{-1} \mathbf{C}' \mathbf{A}^{-1} & \mathbf{I}_{(r)} - (\mathbf{C}' \mathbf{A}^{-1} \mathbf{C})^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{I}_0^{-1} - \mathbf{I}_0^{-1} \mathbf{C} (\mathbf{C}' \mathbf{I}_0^{-1} \mathbf{C})^{-1} \mathbf{C}' \mathbf{I}_0^{-1} & \mathbf{I}_0^{-1} \mathbf{C} (\mathbf{C}' \mathbf{I}_0^{-1} \mathbf{C})^{-1} \\ (\mathbf{C}' \mathbf{I}_0^{-1} \mathbf{C})^{-1} \mathbf{C}' \mathbf{I}_0^{-1} & -(\mathbf{C}' \mathbf{I}_0^{-1} \mathbf{C})^{-1} \end{pmatrix}. \end{aligned}$$

Proof. An alternative direct proof of Corollary S2 is given as follows. First, we derive the equality of the second diagonal blocks of the above equation. Since

$$\begin{aligned}
\mathbf{C}'\mathbf{A}^{-1}\mathbf{C} &= \mathbf{C}'(\mathbf{I}_0 + \mathbf{C}\mathbf{C}')^{-1}\mathbf{C} \\
&= \mathbf{C}'\{\mathbf{I}_0^{-1} - \mathbf{I}_0^{-1}\mathbf{C}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C} + \mathbf{I}_{(r)})^{-1}\mathbf{C}'\mathbf{I}_0^{-1}\}\mathbf{C} \\
&= \mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C} - \mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C} + \mathbf{I}_{(r)})^{-1}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C} + \mathbf{I}_{(r)} - \mathbf{I}_{(r)}) \\
&= \mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C} + \mathbf{I}_{(r)})^{-1},
\end{aligned}$$

$$\begin{aligned}
\mathbf{I}_{(r)} - (\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1} &= \mathbf{I}_{(r)} - (\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C} + \mathbf{I}_{(r)})(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1} \\
&= -(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}.
\end{aligned}$$

The equality of the lower-left (or upper-right) blocks is given as follows.

$$\begin{aligned}
&(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1} \\
&= \{\mathbf{I}_{(r)} + (\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\}\mathbf{C}'\{\mathbf{I}_0^{-1} - \mathbf{I}_0^{-1}\mathbf{C}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C} + \mathbf{I}_{(r)})^{-1}\mathbf{C}'\mathbf{I}_0^{-1}\} \\
&= \{\mathbf{I}_{(r)} + (\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\}[\mathbf{C}'\mathbf{I}_0^{-1} - \{\mathbf{I}_{(r)} - (\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C} + \mathbf{I}_{(r)})^{-1}\}\mathbf{C}'\mathbf{I}_0^{-1}] \\
&= \{\mathbf{I}_{(r)} + (\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C} + \mathbf{I}_{(r)})^{-1}\mathbf{C}'\mathbf{I}_0^{-1} \\
&= \{\mathbf{I}_{(r)} + (\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\}[(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1} \\
&\quad - (\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\{(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1} + \mathbf{I}_{(r)}\}^{-1}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}] \mathbf{C}'\mathbf{I}_0^{-1} \\
&= \{\mathbf{I}_{(r)} + (\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\}\{(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1} + \mathbf{I}_{(r)}\}^{-1}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{I}_0^{-1} \\
&= (\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{I}_0^{-1}.
\end{aligned}$$

The equality of the first diagonal blocks is given as follows.

$$\begin{aligned}
&\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{A}^{-1} \\
&= \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{C}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{I}_0^{-1} \\
&= \mathbf{I}_0^{-1} - \mathbf{I}_0^{-1}\mathbf{C}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C} + \mathbf{I}_{(r)})^{-1}\mathbf{C}'\mathbf{I}_0^{-1} \\
&\quad - \{\mathbf{I}_0^{-1} - \mathbf{I}_0^{-1}\mathbf{C}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C} + \mathbf{I}_{(r)})^{-1}\mathbf{C}'\mathbf{I}_0^{-1}\}\mathbf{C}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{I}_0^{-1} \\
&= \mathbf{I}_0^{-1} - \mathbf{I}_0^{-1}\mathbf{C}(\mathbf{C}'\mathbf{I}_0^{-1}\mathbf{C})^{-1}\mathbf{C}'\mathbf{I}_0^{-1}. \quad \text{Q.E.D.}
\end{aligned}$$

A different expression of the result of Lemma S3 when \mathbf{B} is non-singular is given from the result before Lemma S3 as

Lemma S4. When \mathbf{B} is non-singular and $\mathbf{A} = \mathbf{B} + \mathbf{D}\mathbf{C}'$ as in Lemma S3,

$$\begin{aligned} \begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} &= \begin{pmatrix} \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{A}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1} \\ (\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{A}^{-1} & \mathbf{I}_{(r)} - (\mathbf{C}'\mathbf{A}^{-1}\mathbf{D})^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{B}^{-1} - \mathbf{B}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{B}^{-1} & \mathbf{B}^{-1}\mathbf{D}(\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1} \\ (\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1}\mathbf{C}'\mathbf{B}^{-1} & -(\mathbf{C}'\mathbf{B}^{-1}\mathbf{D})^{-1} \end{pmatrix}. \end{aligned}$$

Let \mathbf{C} and \mathbf{D} in Lemmas S2, S3 and S4 be partitioned as $\mathbf{C} = (\mathbf{C}_{(\cdot,1)} \quad \mathbf{C}_{(\cdot,2)})$ and $\mathbf{D} = (\mathbf{D}_{(\cdot,1)} \quad \mathbf{D}_{(\cdot,2)})$, where $\mathbf{C}_{(\cdot,1)}$ and $\mathbf{D}_{(\cdot,1)}$ are $q \times s$ submatrices, and $\mathbf{C}_{(\cdot,2)}$ and $\mathbf{D}_{(\cdot,2)}$ are $q \times t$ submatrices with $s + t = r$ after possible simultaneous permutations of the columns of \mathbf{C} and \mathbf{D} . Assume that $\mathbf{B} + \mathbf{D}_{(\cdot,2)}\mathbf{C}_{(\cdot,2)}'$ is non-singular, where \mathbf{B} may or may not be singular. The quantity t ($0 < t < r$) is not necessarily the minimum value to yield the non-singular $\mathbf{B} + \mathbf{D}_{(\cdot,2)}\mathbf{C}_{(\cdot,2)}'$ when \mathbf{B} is singular.

Lemma S5. The first and second row-wise blocks of $\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$ and

$\begin{pmatrix} \mathbf{B} + \mathbf{D}_{(\cdot,2)}\mathbf{C}_{(\cdot,2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$ corresponding to the first q rows and the following s rows are equal. Similarly, the first and second column-wise blocks of the matrices corresponding to the first q columns and the following s columns are equal.

Proof. From the identity,

$$\left\{ \begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O} \\ \mathbf{C}_{(\cdot,2)}' & \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix} \right\}^{-1} = \begin{pmatrix} \mathbf{B} + \mathbf{D}_{(\cdot,2)}\mathbf{C}_{(\cdot,2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1},$$

$$\text{we have } \begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O} \\ \mathbf{C}_{(\cdot,2)}' & \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix} \begin{pmatrix} \mathbf{B} + \mathbf{D}_{(\cdot,2)}\mathbf{C}_{(\cdot,2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$$

Similarly, from the identity

$$\left\{ \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} & \mathbf{D}_{(\cdot,2)} \\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix} \begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix} \right\}^{-1} = \begin{pmatrix} \mathbf{B} + \mathbf{D}_{(\cdot,2)} \mathbf{C}_{(\cdot,2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1},$$

using Lemma S1 we have

$$\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} & -\mathbf{D}_{(\cdot,2)} \\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix} = \begin{pmatrix} \mathbf{B} + \mathbf{D}_{(\cdot,2)} \mathbf{C}_{(\cdot,2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$$

The above results show the required equalities. Q.E.D.

Define $\mathbf{A}_{(2)} = \mathbf{B} + \mathbf{D}_{(\cdot,2)} \mathbf{C}_{(\cdot,2)}'$. Then, using Lemma S5, we have

Theorem S2. *The three submatrices other than the second diagonal block on the right-hand side of the identity*

$$\begin{pmatrix} \mathbf{B} + \mathbf{D}_{(\cdot,2)} \mathbf{C}_{(\cdot,2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}_{(2)}^{-1} - \mathbf{A}_{(2)}^{-1} \mathbf{D} (\mathbf{C}' \mathbf{A}_{(2)}^{-1} \mathbf{D})^{-1} \mathbf{C}' & \mathbf{A}_{(2)}^{-1} \mathbf{D} (\mathbf{C}' \mathbf{A}_{(2)}^{-1} \mathbf{D})^{-1} \\ (\mathbf{C}' \mathbf{A}_{(2)}^{-1} \mathbf{D})^{-1} \mathbf{C}' \mathbf{A}_{(2)}^{-1} & -(\mathbf{C}' \mathbf{A}_{(2)}^{-1} \mathbf{D})^{-1} \end{pmatrix}$$

are unchanged irrespective of the choice of $\mathbf{C}_{(\cdot,2)}$ and $\mathbf{D}_{(\cdot,2)}$.

Theorem S3. *Theorem S2 holds even if $\mathbf{B} + \mathbf{D}_{(\cdot,2)} \mathbf{C}_{(\cdot,2)}'$ is replaced by $\mathbf{B} + \mathbf{D}_{(\cdot,2)} \mathbf{E} \mathbf{C}_{(\cdot,2)}'$, where \mathbf{E} is a non-singular $t \times t$ matrix.*

Proof. From the identity

$$\left\{ \begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O} \\ \mathbf{E} \mathbf{C}_{(\cdot,2)}' & \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix} \right\}^{-1} = \begin{pmatrix} \mathbf{B} + \mathbf{D}_{(\cdot,2)} \mathbf{E} \mathbf{C}_{(\cdot,2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}, \text{ we obtain}$$

$$\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O} \\ \mathbf{E} \mathbf{C}_{(\cdot,2)}' & \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix} \begin{pmatrix} \mathbf{B} + \mathbf{D}_{(\cdot,2)} \mathbf{E} \mathbf{C}_{(\cdot,2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}. \text{ Similarly,}$$

$$\left\{ \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} & \mathbf{D}_{(\cdot,2)} \mathbf{E} \\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix} \begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix} \right\}^{-1} = \begin{pmatrix} \mathbf{B} + \mathbf{D}_{(\cdot,2)} \mathbf{E} \mathbf{C}_{(\cdot,2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} \text{ gives}$$

$$\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{B} + \mathbf{D}_{(\cdot,2)} \mathbf{E} \mathbf{C}_{(\cdot,2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} & \mathbf{D}_{(\cdot,2)} \mathbf{E} \\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix}. \text{ These two results}$$

show that $\begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$ and $\begin{pmatrix} \mathbf{B} + \mathbf{D}_{(\cdot,2)} \mathbf{E} \mathbf{C}_{(\cdot,2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1}$ are the same except their second diagonal blocks. Q.E.D.

Crowder's (1984, Lemma 6) result for the augmented matrix derived in a different way is a special case of Theorem S3 when $\mathbf{B} = \mathbf{I}_0$ and

$\mathbf{D} = \mathbf{C} = -\mathbf{H}_0$. Note that the arbitrariness of \mathbf{E} corresponds to the arbitrary expression of the restriction $\mathbf{E}\mathbf{h} = \mathbf{0}$ with \mathbf{E} being non-singular.

Using Lemmas S1, S5 and Theorem S2, we have

Corollary S3. *When \mathbf{B} is possibly singular,*

$$\begin{aligned} \begin{pmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} &= \begin{pmatrix} \mathbf{I}_{(q)} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{(s)} & \mathbf{O} \\ \mathbf{C}_{(\cdot,2)}' & \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix} \begin{pmatrix} \mathbf{B} + \mathbf{D}_{(\cdot,2)} \mathbf{C}_{(\cdot,2)}' & \mathbf{D} \\ \mathbf{C}' & \mathbf{O} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \mathbf{A}_{(2)}^{-1} - \mathbf{A}_{(2)}^{-1} \mathbf{D} (\mathbf{C}' \mathbf{A}_{(2)}^{-1} \mathbf{D})^{-1} \mathbf{C}' & \mathbf{A}_{(2)}^{-1} \mathbf{D} (\mathbf{C}' \mathbf{A}_{(2)}^{-1} \mathbf{D})^{-1} \\ (\mathbf{C}' \mathbf{A}_{(2)}^{-1} \mathbf{D})^{-1} \mathbf{C}' \mathbf{A}_{(2)}^{-1} & \begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix} - (\mathbf{C}' \mathbf{A}_{(2)}^{-1} \mathbf{D})^{-1} \end{pmatrix}. \end{aligned}$$

Equating the results of Lemma S3 and Corollary S3, we find that

$$\begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix} - (\mathbf{C}' \mathbf{A}_{(2)}^{-1} \mathbf{D})^{-1} = \mathbf{I}_{(r)} - (\mathbf{C}' \mathbf{A}^{-1} \mathbf{D})^{-1} \text{ or equivalently}$$

$$(\mathbf{C}' \mathbf{A}^{-1} \mathbf{D})^{-1} - (\mathbf{C}' \mathbf{A}_{(2)}^{-1} \mathbf{D})^{-1} = \begin{pmatrix} \mathbf{I}_{(s)} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}. \text{ When } \mathbf{B} \text{ is non-singular.}$$

$$\begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{(t)} \end{pmatrix} - (\mathbf{C}' \mathbf{A}_{(2)}^{-1} \mathbf{D})^{-1} = -(\mathbf{C}' \mathbf{B}^{-1} \mathbf{D})^{-1}.$$

3. Applications of the general formulas to Example 2.1 (Example 1.2)

3.1 Non-studentized estimators

$$\begin{aligned} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} &\equiv n^{-1} \left(\frac{\sum_{i=1}^{n_1} x_{1i} - n_1 \theta_{01}}{\theta_{01}(1-\theta_{01})}, \frac{\sum_{i=1}^{n_2} x_{2i} - n_2 \theta_{02}}{\theta_{02}(1-\theta_{02})} \right)' \\ &= n^{-1} \left(\frac{m_{11} - n_1 \theta_{01}}{\theta_{01}(1-\theta_{01})}, \frac{m_{21} - n_2 \theta_{02}}{\theta_{02}(1-\theta_{02})} \right)', \\ \frac{\partial^2 \bar{l}}{\partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0'} &= n^{-1} \begin{bmatrix} \frac{-n_1}{\theta_{01}(1-\theta_{01})} - \frac{m_{11} - n_1 \theta_{01}}{\{\theta_{01}(1-\theta_{01})\}^2} (1-2\theta_{01}) & 0 \\ 0 & \frac{-n_2}{\theta_{02}(1-\theta_{02})} - \frac{m_{21} - n_2 \theta_{02}}{\{\theta_{02}(1-\theta_{02})\}^2} (1-2\theta_{02}) \end{bmatrix}, \\ E_{\theta} \left(\frac{\partial^2 \bar{l}}{\partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0'} \right) &= -n^{-1} \begin{bmatrix} \frac{n_1}{\theta_{01}(1-\theta_{01})} & 0 \\ 0 & \frac{n_2}{\theta_{02}(1-\theta_{02})} \end{bmatrix} \\ &= - \begin{bmatrix} \frac{c_1}{\theta_{01}(1-\theta_{01})} & 0 \\ 0 & \frac{c_2}{\theta_{02}(1-\theta_{02})} \end{bmatrix} \\ &= -\mathbf{I}_0 \quad (c_1 = n_1/n = O(1), c_2 = n_2/n = O(1)), \end{aligned}$$

$$\begin{aligned}
\mathbf{M} &\equiv \frac{\partial^2 \bar{l}}{\partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0'} - E_{\theta} \left(\frac{\partial^2 \bar{l}}{\partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0'} \right) \\
&= -n^{-1} \begin{bmatrix} \frac{m_{11} - n_1 \theta_{01}}{\{\theta_{01}(1 - \theta_{01})\}^2} (1 - 2\theta_{01}) & 0 \\ 0 & \frac{m_{21} - n_2 \theta_{02}}{\{\theta_{02}(1 - \theta_{02})\}^2} (1 - 2\theta_{02}) \end{bmatrix}, \\
E_{\theta} \left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right) &= n^{-1} \begin{pmatrix} \frac{c_1}{\theta_{01}(1 - \theta_{01})} & 0 \\ 0 & \frac{c_2}{\theta_{02}(1 - \theta_{02})} \end{pmatrix} = n^{-1} \mathbf{I}_0, \\
E_{\theta} \left((\mathbf{M})_{11} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right) &= n^{-1} \left(-\frac{c_1}{\{\theta_{01}(1 - \theta_{01})\}^2} (1 - 2\theta_{01}), 0 \right), \\
E_{\theta} \left((\mathbf{M})_{22} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right) &= n^{-1} \left(0, -\frac{c_2}{\{\theta_{02}(1 - \theta_{02})\}^2} (1 - 2\theta_{02}) \right), \\
E_{\theta} \left((\mathbf{M})_{12} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right) &= E_{\theta} \left((\mathbf{M})_{21} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right) = \mathbf{0}', \\
\boldsymbol{\Lambda}_0^{*-1} &= \begin{pmatrix} -\mathbf{I}_0 & \mathbf{H}_0 \\ \mathbf{H}_0' & \mathbf{0} \end{pmatrix}^{-1}.
\end{aligned}$$

Since $\theta_{01} = \theta_{02}$, define $\theta_0 \equiv (\boldsymbol{\theta}_0)_1 = (\boldsymbol{\theta}_0)_2 = \theta_{01} = \theta_{02}$. Then, using

$$\mathbf{I}_0 = \frac{1}{\theta_0(1 - \theta_0)} \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \text{ and } \mathbf{I}_0^{-1} = \theta_0(1 - \theta_0) \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix},$$

$$\begin{aligned}
\Lambda_0^{*-1} &= \begin{bmatrix} -\mathbf{I}_0^{-1} + \mathbf{I}_0^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left\{ (1, -1) \mathbf{I}_0^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}^{-1} (1, -1) \mathbf{I}_0^{-1} \text{ sym.} \\ \left\{ (1, -1) \mathbf{I}_0^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}^{-1} (1, -1) \mathbf{I}_0^{-1} & \quad \left\{ (1, -1) \mathbf{I}_0^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \theta_0(1-\theta_0) \left\{ -\begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix} + c_1 c_2 \begin{pmatrix} c_1^{-1} \\ -c_2^{-1} \end{pmatrix} (c_1^{-1}, -c_2^{-1}) \right\} \text{ sym.} \\ c_1 c_2 (c_1^{-1}, -c_2^{-1}) & \quad \frac{c_1 c_2}{\theta_0(1-\theta_0)} \end{bmatrix}, \\
&= \begin{bmatrix} \theta_0(1-\theta_0) \begin{pmatrix} -\frac{1-c_2}{c_1} & -1 \\ -1 & -\frac{1-c_1}{c_2} \end{pmatrix} & \begin{pmatrix} c_2 \\ -c_1 \end{pmatrix} \\ (c_2, -c_1) & \frac{c_1 c_2}{\theta_0(1-\theta_0)} \end{bmatrix} \\
&= \begin{bmatrix} -\theta_0(1-\theta_0) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} c_2 \\ -c_1 \end{pmatrix} \\ (c_2, -c_1) & \frac{c_1 c_2}{\theta_0(1-\theta_0)} \end{bmatrix}, \\
\mathbf{q}_0^* &= \begin{pmatrix} \frac{k}{\theta_{01}} - \frac{k}{1-\theta_{01}}, \frac{k}{\theta_{02}} - \frac{k}{1-\theta_{02}} \end{pmatrix}' = k \begin{pmatrix} \frac{1-2\theta_{01}}{\theta_{01}(1-\theta_{01})}, \frac{1-2\theta_{02}}{\theta_{02}(1-\theta_{02})} \end{pmatrix}' \\
&= \frac{k(1-2\theta_0)}{\theta_0(1-\theta_0)} (1, 1)'.
\end{aligned}$$

$$\Lambda_{\text{W}}^{(1)} \mathbf{I}_0^{(1)} = \Lambda_{\text{ML}}^{(1)} \mathbf{I}_0^{(1)} = -\Lambda_0^{(\cdot 1)} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} = - \begin{bmatrix} -\theta_0(1-\theta_0) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ (c_2, -c_1) \end{bmatrix} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0},$$

where $\Lambda_{\text{W}}^{(1)} = \Lambda_{\text{ML}}^{(1)} = -\Lambda_0^{(\cdot 1)} = -(\Lambda_0^{(11)} \Lambda_0^{(21)})'$, $\mathbf{I}_0^{(1)} = \partial \bar{l} / \partial \boldsymbol{\theta}_0$.

$$\begin{aligned}
& -n^{-1} \begin{pmatrix} \mathbf{\Lambda}_0^{(11)} \\ \mathbf{\Lambda}_0^{(21)} \end{pmatrix} \mathbf{q}_0^* = n^{-1} \begin{bmatrix} \theta_0(1-\theta_0) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ -(c_2, -c_1) \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{k(1-2\theta_0)}{\theta_0(1-\theta_0)} \\
& = n^{-1} \begin{bmatrix} 2 \\ 2 \\ (c_1 - c_2) / \{\theta_0(1-\theta_0)\} \end{bmatrix} k(1-2\theta_0),
\end{aligned}$$

where $\mathbf{\Lambda}_0^{(11)} \mathbf{q}_0^* = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} 2k(1-2\theta_0)$.

For nonzero elements of $\mathbf{J}_0^{(3)}$,

$$\begin{aligned}
& (\mathbf{J}_0^{(3)})_{(\theta_1, \theta_1 \theta_1)} \\
& = n^{-1} \left\{ \frac{n_1 2(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2} - (m_{11} - n_1 \theta_0) \left(-\frac{2(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} - \frac{2}{\{\theta_0(1-\theta_0)\}^2} \right) \right\} \\
& = 2n^{-1} \left\{ \frac{n_1(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2} + (m_{11} - n_1 \theta_0) \left(\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} + \frac{1}{\{\theta_0(1-\theta_0)\}^2} \right) \right\},
\end{aligned}$$

$$\mathbf{E}_\theta \{(\mathbf{J}_0^{(3)})_{(\theta_1, \theta_1 \theta_1)}\} = 2c_1 \frac{1-2\theta_0}{\{\theta_0(1-\theta_0)\}^2},$$

$$\begin{aligned}
& \{\mathbf{J}_0^{(3)} - \mathbf{E}_\theta(\mathbf{J}_0^{(3)})\}_{(\theta_1, \theta_1 \theta_1)} \\
& = 2n^{-1} (m_{11} - n_1 \theta_0) \left(\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} + \frac{1}{\{\theta_0(1-\theta_0)\}^2} \right).
\end{aligned}$$

Similarly,

$$\mathbf{E}_\theta \{(\mathbf{J}_0^{(3)})_{(\theta_2, \theta_2 \theta_2)}\} = 2c_2 \frac{1-2\theta_0}{\{\theta_0(1-\theta_0)\}^2},$$

$$\begin{aligned}
& \{\mathbf{J}_0^{(3)} - \mathbf{E}_\theta(\mathbf{J}_0^{(3)})\}_{(\theta_2, \theta_2 \theta_2)} \\
& = 2n^{-1} (m_{21} - n_2 \theta_0) \left(\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} + \frac{1}{\{\theta_0(1-\theta_0)\}^2} \right).
\end{aligned}$$

Nonzero elements of $\mathbf{J}_0^{(4)}$ are

$$\begin{aligned} \mathbf{E}_\theta \{(\mathbf{J}_0^{(4)})_{(\theta_1, \theta_1 \theta_1 \theta_1)}\} &= 2c_1 \left(-\frac{3(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} - \frac{3}{\{\theta_0(1-\theta_0)\}^2} \right) \\ &= -6c_1 \left(\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} + \frac{1}{\{\theta_0(1-\theta_0)\}^2} \right), \end{aligned}$$

$$\mathbf{E}_\theta \{(\mathbf{J}_0^{(4)})_{(\theta_2, \theta_2 \theta_2 \theta_2)}\} = -6c_2 \left(\frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} + \frac{1}{\{\theta_0(1-\theta_0)\}^2} \right).$$

$$\Lambda_{\mathbf{W}}^{(2)} \mathbf{I}_0^{(2)} = \Lambda_{\text{ML}}^{(2)} \mathbf{I}_0^{(2)} = (\Lambda_{\text{ML}}^{(2-1)} \quad \Lambda_{\text{ML}}^{(2-2)}) \left\{ \mathbf{v}(\mathbf{M})' \otimes \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'}, \left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right)^{\langle 2 \rangle} \right\}',$$

where

$$\begin{aligned} (\Lambda_{\text{ML}}^{(2-1)})_{\cdot(ijk^*)} &= \sum_{(ij)}^2 (\Lambda_0^{(\cdot 1)})_{\cdot i} \frac{2-\delta_{ij}}{2} (\Lambda_0^{(11)})_{jk^*} \\ &= \sum_{(ij)}^2 \left[\begin{array}{c} -\theta_0(1-\theta_0) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ (c_2, -c_1) \end{array} \right]_{\cdot i} \frac{2-\delta_{ij}}{2} \{-\theta_0(1-\theta_0)\} \\ &= \left[\begin{array}{c} \{\theta_0(1-\theta_0)\}^2 (2-\delta_{ij}) \\ -\sum_{(ij)}^2 \theta_0(1-\theta_0)(c_2, -c_1)_i (2-\delta_{ij}) / 2 \end{array} \right] (1 \leq i \leq j \leq 2; k^* = 1, 2), \end{aligned}$$

$$\begin{aligned}
(\mathbf{\Lambda}_{\text{ML}}^{(2-2)})_{\cdot(ij)} &= -\frac{1}{2} \mathbf{\Lambda}_0^{*-1} \begin{pmatrix} \mathbf{E}_\theta(\mathbf{J}_0^{(3)}) \\ \mathbf{0} \end{pmatrix} \{(\mathbf{\Lambda}_0^{(11)})_{\cdot i} \otimes (\mathbf{\Lambda}_0^{(11)})_{\cdot j}\} \\
&= -\frac{1}{2} \mathbf{\Lambda}_0^{(\cdot 1)} \mathbf{E}_\theta(\mathbf{J}_0^{(3)}) \{(\mathbf{\Lambda}_0^{(11)})_{\cdot i} \otimes (\mathbf{\Lambda}_0^{(11)})_{\cdot j}\} \\
&= -\frac{1}{2} \begin{bmatrix} -\theta_0(1-\theta_0) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ (c_2, -c_1) \end{bmatrix} \mathbf{E}_\theta(\mathbf{J}_0^{(3)}) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \{\theta_0(1-\theta_0)\}^2 \\
&= -\begin{bmatrix} -\theta_0(1-\theta_0) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ (c_2, -c_1) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} (1-2\theta_0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \theta_0(1-\theta_0)(1-2\theta_0) \\
&(i, j = 1, \dots, q).
\end{aligned}$$

$$\mathbf{\Lambda}_{\text{W}}^{(3)} \mathbf{I}_0^{(3)} = \mathbf{\Lambda}_{\text{ML}}^{(3)} \mathbf{I}_0^{(3)} + \mathbf{\Lambda}_{\text{W}}^{(\Delta)} n^{-1} \mathbf{I}_0^{(\text{W})},$$

$$\mathbf{\Lambda}_{\text{ML}}^{(3)} \mathbf{I}_0^{(3)}$$

$$\begin{aligned}
&= (\mathbf{\Lambda}_{\text{ML}}^{(3-1)} \mathbf{\Lambda}_{\text{ML}}^{(3-2)} \mathbf{\Lambda}_{\text{ML}}^{(3-3)} \mathbf{\Lambda}_{\text{ML}}^{(3-4)}) \left\{ \mathbf{v}(\mathbf{M})'^{\langle 2 \rangle} \otimes \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'}, \mathbf{v}(\mathbf{M})' \otimes \left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right)^{\langle 2 \rangle} \right. \\
&\quad \left. \mathbf{vec}((\mathbf{J}_0^{(3)} - \mathbf{E}_{\text{T}}(\mathbf{J}_0^{(3)}))' \otimes \left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right)^{\langle 2 \rangle}, \left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right)^{\langle 3 \rangle} \right\}',
\end{aligned}$$

$$\Lambda_W^{(\Delta)} n^{-1} \mathbf{I}_0^{(W)} = (\Lambda_W^{(\Delta-1)} \Lambda_W^{(\Delta-2)}) n^{-1} \left\{ \mathbf{v}(\mathbf{M})', \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right\}',$$

$$(\Lambda_{ML}^{(3-1)})_{\cdot(ijk^*l^*m)} = - \sum_{(ij)}^2 \sum_{(k^*l^*)}^2 (\Lambda_0^{(\cdot 1)})_{\cdot i} \frac{2 - \delta_{ij}}{2} (\Lambda_0^{(11)})_{jk^*} \frac{2 - \delta_{k^*l^*}}{2} (\Lambda_0^{(11)})_{l^*m}$$

$$= - \sum_{(ij)}^2 \sum_{(k^*l^*)}^2 \left[\begin{array}{c} -\theta_0(1-\theta_0) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ (c_2, -c_1) \end{array} \right]_{\cdot i} \frac{(2 - \delta_{ij})(2 - \delta_{k^*l^*})}{4} \{\theta_0(1 - \theta_0)\}^2$$

$$= \left[\begin{array}{c} \{\theta_0(1 - \theta_0)\}^3 (2 - \delta_{ij}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ - \sum_{(ij)}^2 (c_2, -c_1)_{\cdot i} \{(2 - \delta_{ij}) / 2\} \{\theta_0(1 - \theta_0)\}^2 \end{array} \right] (2 - \delta_{k^*l^*})$$

$$(1 \leq i \leq j \leq 2; 1 \leq k^* \leq l^* \leq 2; m = 1, 2),$$

$$(\Lambda_{ML}^{(3-2)})_{\cdot(ijk^*l^*)}$$

$$= \sum_{(ij)}^2 (\Lambda_0^{(\cdot 1)})_{\cdot i} \frac{2 - \delta_{ij}}{2} \left(\frac{1}{2} \Lambda_0^{(11)} \mathbf{E}_\theta(\mathbf{J}_0^{(3)}) \{(\Lambda_0^{(11)})_{\cdot k^*} \otimes (\Lambda_0^{(11)})_{\cdot l^*}\} \right)_j$$

$$+ \Lambda_0^{(\cdot 1)} \mathbf{E}_\theta(\mathbf{J}_0^{(3)}) \left[(\Lambda_0^{(11)})_{\cdot k^*} \otimes \sum_{(ij)}^2 \left\{ (\Lambda_0^{(11)})_{\cdot i} \frac{2 - \delta_{ij}}{2} (\Lambda_0^{(11)})_{jl^*} \right\} \right]$$

$$= \sum_{(ij)}^2 (\Lambda_0^{(\cdot 1)})_{\cdot i} \frac{2 - \delta_{ij}}{2} (\Lambda_0^{(11)})_{j\cdot} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} (1 - 2\theta_0)$$

$$- \Lambda_0^{(\cdot 1)} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} 2\theta_0(1 - \theta_0)(1 - 2\theta_0)(2 - \delta_{ij})$$

$$= \sum_{(ij)}^2 \left(\begin{array}{c} -\theta_0(1-\theta_0) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ (c_2, -c_1) \end{array} \right)_{\cdot i} \frac{2-\delta_{ij}}{2} (-1)\theta_0(1-\theta_0)(1-2\theta_0)$$

$$- \left(\begin{array}{c} -\theta_0(1-\theta_0) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ (c_2, -c_1) \end{array} \right) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} 2\theta_0(1-\theta_0)(1-2\theta_0)(2-\delta_{ij})$$

$$= \left(\begin{array}{c} \begin{pmatrix} 1 \\ 1 \end{pmatrix} 3(2-\delta_{ij}) \{\theta_0(1-\theta_0)\}^2 (1-2\theta_0) \\ - \sum_{(ij)}^2 (c_2, -c_1)_{\cdot i} \frac{2-\delta_{ij}}{2} \theta_0(1-\theta_0)(1-2\theta_0) \end{array} \right)$$

$$(1 \leq i \leq j \leq 2; k^*, l^* = 1, 2),$$

$$(\Lambda_{\text{ML}}^{(3-3)})_{\cdot (ijk^*l^*m)} = -\frac{1}{2} (\Lambda_0^{(\cdot 1)})_{\cdot i} (\Lambda_0^{(11)})_{j l^*} (\Lambda_0^{(11)})_{k^* m}$$

$$= -\frac{1}{2} \left(\begin{array}{c} -\theta_0(1-\theta_0) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ (c_2, -c_1) \end{array} \right)_{\cdot i} \{\theta_0(1-\theta_0)\}^2$$

$$= \frac{1}{2} \left(\begin{array}{c} \theta_0(1-\theta_0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ -(c_2, -c_1) \end{array} \right)_{\cdot i} \{\theta_0(1-\theta_0)\}^2$$

$$(i, j, k^*, l^*, m = 1, 2).$$

$$(\Lambda_{\text{ML}}^{(3-4)})_{\cdot (ijk^*)}$$

$$= -\frac{1}{2} \Lambda_0^{(\cdot 1)} E_{\theta}(\mathbf{J}_0^{(3)}) \left[(\Lambda_0^{(11)})_{\cdot i} \otimes \left\{ \Lambda_0^{(11)} E_{\theta}(\mathbf{J}_0^{(3)}) \{ (\Lambda_0^{(11)})_{\cdot j} \otimes \Lambda_0^{(11)}_{\cdot k} \} \right\} \right]$$

$$+ \frac{1}{6} \Lambda_0^{(\cdot 1)} E_{\theta}(\mathbf{J}_0^{(4)}) \{ (\Lambda_0^{(11)})_{\cdot i} \otimes (\Lambda_0^{(11)})_{\cdot j} \otimes \Lambda_0^{(11)}_{\cdot k^*} \},$$

where the first term is

$$\begin{aligned}
& -\frac{1}{2}\mathbf{\Lambda}_0^{(\cdot 1)}\mathbf{E}_\theta(\mathbf{J}_0^{(3)})\left[\left(\mathbf{\Lambda}_0^{(11)}\right)_{\cdot i}\otimes\left\{\mathbf{\Lambda}_0^{(11)}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}2(1-2\theta_0)\right\}\right] \\
& =-\frac{1}{2}\mathbf{\Lambda}_0^{(\cdot 1)}\mathbf{E}_\theta(\mathbf{J}_0^{(3)})\left\{\theta_0(1-\theta_0)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 1 \\ 1 \end{pmatrix}\theta_0(1-\theta_0)2(1-2\theta_0)\right\} \\
& =-\frac{1}{2}\mathbf{\Lambda}_0^{(\cdot 1)}4\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}(1-2\theta_0)^2 \\
& =2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\theta_0(1-\theta_0)(1-2\theta_0)^2
\end{aligned}$$

and the second term is

$$\begin{aligned}
& \frac{1}{6}\mathbf{\Lambda}_0^{(\cdot 1)}6\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\{(1-2\theta_0)^2+\theta_0(1-\theta_0)\} \\
& =-\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}[\theta_0(1-\theta_0)(1-2\theta_0)^2+\{\theta_0(1-\theta_0)\}^2].
\end{aligned}$$

Then,

$$\left(\mathbf{\Lambda}_{\text{ML}}^{(3-4)}\right)_{\cdot(ijk^*)}=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\left[\theta_0(1-\theta_0)(1-2\theta_0)^2-\{\theta_0(1-\theta_0)\}^2\right].$$

$$\begin{aligned}
\left(\mathbf{\Lambda}_{\text{ML}}^{(\Delta-1)}\right)_{\cdot(ij)} & =\sum_{(ij)}^2\left(\mathbf{\Lambda}_0^{(\cdot 1)}\right)_{\cdot i}\frac{2-\delta_{ij}}{2}\left(\mathbf{\Lambda}_0^{(11)}\mathbf{q}_0^*\right)_j \\
& =\sum_{(ij)}^2\left(\mathbf{\Lambda}_0^{(\cdot 1)}\right)_{\cdot i}\frac{2-\delta_{ij}}{2}\{-2k(1-2\theta_0)\} \\
& =-\sum_{(ij)}^2\begin{pmatrix} -\theta_0(1-\theta_0)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ (c_2, -c_1) \end{pmatrix}_{\cdot i}(2-\delta_{ij})k(1-2\theta_0) \\
& =\begin{pmatrix} \theta_0(1-\theta_0)2(2-\delta_{ij}) \\ -\sum_{(ij)}^2(c_2, -c_1)_i(2-\delta_{ij}) \end{pmatrix}k(1-2\theta_0)
\end{aligned}$$

$$(1 \leq i \leq j \leq 2),$$

where $\mathbf{q}_0^* = \frac{k(1-2\theta_0)}{\theta_0(1-\theta_0)}(1,1)'$, $\Lambda_0^{(\cdot 1)} = \begin{pmatrix} -\theta_0(1-\theta_0) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ (c_2, -c_1) \end{pmatrix}$ and consequently,

$$\Lambda_0^{(11)} \mathbf{q}_0^* = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} 2k(1-2\theta_0) \text{ is used.}$$

$$(\Lambda_{\text{ML}}^{(\Delta-2)})_{\cdot i} = -\Lambda_0^{(\cdot 1)} \mathbf{E}_\theta(\mathbf{J}_0^{(3)}) \{(\Lambda_0^{(11)})_{\cdot i} \otimes (\Lambda_0^{(11)} \mathbf{q}_0^*)\} + \Lambda_0^{(\cdot 1)} \frac{\partial \mathbf{q}_0^*}{\partial \theta_0} (\Lambda_0^{(11)})_{\cdot i}$$

$$= -\Lambda_0^{(\cdot 1)} \mathbf{E}_\theta(\mathbf{J}_0^{(3)}) \left\{ \begin{pmatrix} -\theta_0(1-\theta_0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ -\begin{pmatrix} 1 \\ 1 \end{pmatrix} 2k(1-2\theta_0) \end{pmatrix} \right\}$$

$$+ \Lambda_0^{(\cdot 1)} \left[-\frac{2}{\theta_0(1-\theta_0)} - \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2} \right] k \mathbf{I}_{(2)} (\Lambda_0^{(11)})_{\cdot i}$$

$$= -\Lambda_0^{(\cdot 1)} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \frac{4k(1-2\theta_0)^2}{\theta_0(1-\theta_0)}$$

$$+ \Lambda_0^{(\cdot 1)} \left[\frac{2}{\theta_0(1-\theta_0)} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2} \right] k \theta_0(1-\theta_0) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= 4k(1-2\theta_0)^2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \left[\frac{2}{\theta_0(1-\theta_0)} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2} \right]$$

$$\times k \theta_0(1-\theta_0) \begin{pmatrix} -\theta_0(1-\theta_0)2 \\ -\theta_0(1-\theta_0)2 \\ c_2 - c_1 \end{pmatrix}$$

$$= k \begin{bmatrix} \{-4\theta_0(1-\theta_0) + 2(1-2\theta_0)^2\} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \left\{ 2 + \frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)} \right\} (c_2 - c_1) \end{bmatrix} \quad (i = 1, 2).$$

3.2 Studentized estimators

3.2.1 $t_{w\theta}$ ($t_{w\theta_1}$ and $t_{w\theta_2}$)

Using θ_{01} and θ_{02} rather than θ_0 for differentiation,

$$\mathbf{I}_0^{*-1} = \{-E_\theta(\mathbf{\Lambda}_0^*)\}^{-1} = \begin{pmatrix} \mathbf{I}_0 & -\mathbf{H}_0 \\ -\mathbf{H}_0' & \mathbf{0} \end{pmatrix}^{-1}$$

$$= \begin{bmatrix} \mathbf{I}_0^{-1} - \mathbf{I}_0^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left\{ (1, -1) \mathbf{I}_0^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}^{-1} & (1, -1) \mathbf{I}_0^{-1} & \text{Sym.} \\ -\left\{ (1, -1) \mathbf{I}_0^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}^{-1} & (1, -1) \mathbf{I}_0^{-1} & -\left\{ (1, -1) \mathbf{I}_0^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}^{-1} \end{bmatrix},$$

where $\mathbf{I}_0^{-1} = \begin{pmatrix} \frac{c_1}{\theta_{01}(1-\theta_{01})} & 0 \\ 0 & \frac{c_2}{\theta_{02}(1-\theta_{02})} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\theta_{01}(1-\theta_{01})}{c_1} & 0 \\ 0 & \frac{\theta_{02}(1-\theta_{02})}{c_2} \end{pmatrix}.$

Then, noting $\mathbf{I}_0^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \theta_{01}(1-\theta_{01})/c_1 \\ -\theta_{02}(1-\theta_{02})/c_2 \end{pmatrix}$ and

$(1, -1) \mathbf{I}_0^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{\theta_{01}(1-\theta_{01})}{c_1} + \frac{\theta_{02}(1-\theta_{02})}{c_2}$, it follows that

$\mathbf{I}_0^{*-1} = \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}' & d \end{pmatrix}$, where

$$\mathbf{A} = \begin{pmatrix} \frac{\theta_{01}(1-\theta_{01})}{c_1} & 0 \\ 0 & \frac{\theta_{02}(1-\theta_{02})}{c_2} \end{pmatrix} - \left(\frac{\theta_{01}(1-\theta_{01})}{c_1} + \frac{\theta_{02}(1-\theta_{02})}{c_2} \right)^{-1}$$

$$\times \begin{pmatrix} \frac{\theta_{01}(1-\theta_{01})}{c_1} \\ -\frac{\theta_{02}(1-\theta_{02})}{c_2} \end{pmatrix} \begin{pmatrix} \frac{\theta_{01}(1-\theta_{01})}{c_1}, & -\frac{\theta_{02}(1-\theta_{02})}{c_2} \end{pmatrix},$$

$$\mathbf{b} = -\left(\frac{\theta_{01}(1-\theta_{01})}{c_1} + \frac{\theta_{02}(1-\theta_{02})}{c_2} \right)^{-1} \begin{pmatrix} \frac{\theta_{01}(1-\theta_{01})}{c_1}, & -\frac{\theta_{02}(1-\theta_{02})}{c_2} \end{pmatrix},$$

$$\text{and } d = - \left(\frac{\theta_{01}(1-\theta_{01})}{c_1} + \frac{\theta_{02}(1-\theta_{02})}{c_2} \right)^{-1}.$$

Since direct differentiation of \mathbf{I}_0^{*-1} with respect to θ_{0k} is tedious, we use the formula of $\partial \mathbf{I}_0^{*-1} / \partial \theta_{0k^*} = -\mathbf{I}_0^{*-1} (\partial \mathbf{I}_0^* / \partial \theta_{0k^*}) \mathbf{I}_0^{*-1}$, where

$$\mathbf{I}_0^* = \begin{pmatrix} \mathbf{I}_0 & -\mathbf{H}_0 \\ -\mathbf{H}_0' & 0 \end{pmatrix} = \begin{pmatrix} \frac{c_1}{\theta_{01}(1-\theta_{01})} & 0 & -1 \\ 0 & \frac{c_2}{\theta_{02}(1-\theta_{02})} & 1 \\ -1 & 1 & 0 \end{pmatrix} \text{ though}$$

$$\mathbf{I}_0^{*-1} = \begin{pmatrix} \theta_0(1-\theta_0) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & -\begin{pmatrix} c_2 \\ -c_1 \end{pmatrix} \\ -(c_2, -c_1) & -\frac{c_1 c_2}{\theta_0(1-\theta_0)} \end{pmatrix} \text{ after evaluation using } \theta_{01} = \theta_{02}$$

looks simple.

$$\begin{aligned} (\mathbf{i}_{\theta_1}^{(1)})_j &= \frac{1}{2} \left(\mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0j}} \mathbf{I}_0^{*-1} \right)_{11} (i_0^{(11)})^{-3/2} \\ &= \frac{1}{2} \{\theta_0(1-\theta_0)\}^2 \frac{-c_j(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2} \{\theta_0(1-\theta_0)\}^{-3/2} \\ &= -\frac{c_j}{2} (1-2\theta_0) \{\theta_0(1-\theta_0)\}^{-3/2} \end{aligned}$$

($j = 1, 2$),

$$(\mathbf{i}_{\theta_2}^{(1)})_j = \frac{1}{2} \left(\mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0j}} \mathbf{I}_0^{*-1} \right)_{22} (i_0^{(11)})^{-3/2} = -\frac{c_j}{2} (1-2\theta_0) \{\theta_0(1-\theta_0)\}^{-3/2}$$

($j = 1, 2$).

That is, $\mathbf{i}_{\theta_1}^{(1)} = \mathbf{i}_{\theta_2}^{(1)}$. Note that

$$\left(\mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0j}} \mathbf{I}_0^{*-1} \right)_{11} = \left(\mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0j}} \mathbf{I}_0^{*-1} \right)_{22} = -c_j(1-2\theta_0) \quad (j=1,2).$$

In $t_{w\theta} \equiv n^{1/2} (\hat{i}_w^{\theta\theta})^{-1/2} (\hat{\theta}_w - \theta_0)$, $\hat{i}_w^{\theta\theta}$ is the first or second diagonal element of \mathbf{I}_0^{*-1} with θ_0 replaced by $\hat{\theta}_w$ and becomes $\hat{\theta}_w(1-\hat{\theta}_w)$. Then, $t_{w\theta} = n^{1/2} \{\hat{\theta}_w(1-\hat{\theta}_w)\}^{-1/2} (\hat{\theta}_w - \theta_0)$, where note that $t_{w\theta}$ is defined using $n^{1/2}$ rather than $n_w^{1/2}$. That is, $\hat{\theta}_w$ and $t_{w\theta}$ reduce to those of a single group proportion with size n and pseudocounts $4k$ in total.

$$\begin{aligned} & (\mathbf{i}_{\theta_1}^{(2)})_{(jk^*)} \\ &= \left(\frac{1}{4} \mathbf{I}_0^{*-1} \frac{\partial^2 \mathbf{I}_0^*}{\partial \theta_{0j} \partial \theta_{0k^*}} \mathbf{I}_0^{*-1} - \frac{1}{2} \mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0j}} \mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0k^*}} \mathbf{I}_0^{*-1} \right)_{11} \{\theta_0(1-\theta_0)\}^{-3/2} \\ &+ \frac{3}{8} \left(\mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0j}} \mathbf{I}_0^{*-1} \right)_{11} \left(\mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0k^*}} \mathbf{I}_0^{*-1} \right)_{11} \{\theta_0(1-\theta_0)\}^{-5/2} \\ &= \left[\delta_{jk^*} \frac{c_j}{2} \left\{ 1 + \frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)} \right\} - \frac{1}{2} c_j c_{k^*} \frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)} \right] \theta_0(1-\theta_0)^{-3/2} \\ &+ \frac{3}{8} c_j c_{k^*} (1-2\theta_0)^2 \{\theta_0(1-\theta_0)\}^{-5/2} \end{aligned}$$

$(j, k^* = 1, 2),$

$$\text{where } \frac{\partial \mathbf{I}_0^*}{\partial \theta_{01}} = \begin{pmatrix} -\frac{c_1(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \frac{\partial \mathbf{I}_0^*}{\partial \theta_{02}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{c_1(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and non-zero second derivatives are

$$\frac{\partial^2 \mathbf{I}_0^*}{(\partial \theta_{01})^2} = \begin{bmatrix} 2c_1 \left(\frac{1}{\{\theta_0(1-\theta_0)\}^2} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} \right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\frac{\partial^2 \mathbf{I}_0^*}{(\partial \theta_{02})^2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2c_2 \left(\frac{1}{\{\theta_0(1-\theta_0)\}^2} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} \right) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$i_0^{11} = i_0^{22} = \theta_0(1-\theta_0), \quad (\mathbf{i}_{\theta_1}^{(2)})_{(jk^*)} = (\mathbf{i}_{\theta_2}^{(2)})_{(jk^*)}.$$

Incidentally,

$$\begin{aligned} \sum_{j=1}^2 \sum_{k^*=1}^2 (\mathbf{i}_{\theta_1}^{(2)})_{(j,k^*)} &= \sum_{j=1}^2 \sum_{k^*=1}^2 (\mathbf{i}_{\theta_2}^{(2)})_{(j,k^*)} \\ &= \left[\frac{c_1 + c_2}{2} \left\{ 1 + \frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)} \right\} - \frac{1}{2} (c_1 + c_2)(c_1 + c_2) \frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)} \right] \{\theta_0(1-\theta_0)\}^{-3/2} \\ &\quad + \frac{3}{8} (c_1 + c_2)(c_1 + c_2)(1-2\theta_0)^2 \{\theta_0(1-\theta_0)\}^{-5/2} \\ &= \frac{1}{2} \{\theta_0(1-\theta_0)\}^{-3/2} + \frac{3}{8} (1-2\theta_0) \{\theta_0(1-\theta_0)\}^{-5/2} \\ &= i_0^{(2)}, \end{aligned}$$

which is a scalar in the case of a single group.

3.2.2 $t_{W\eta}$

$$\text{Expand } (-\hat{i}_W^{\eta\eta})^{-1/2} \text{ about } (-i_0^{\eta\eta})^{-1/2} \left(= \left\{ \frac{c_1 c_2}{\theta_0(1-\theta_0)} \right\}^{-1/2} \right),$$

$$\text{where } \text{avar}(n^{-1} \hat{\eta}_W) = n^{-1} (-\mathbf{I}_0^{*-1})_{\eta\eta} = n^{-1} \frac{c_1 c_2}{\theta_0(1-\theta_0)}.$$

$$\begin{aligned}
(-\hat{i}_W^{\eta\eta})^{-1/2} &= (-i_0^{\eta\eta})^{-1/2} - \frac{1}{2} \sum_{j=1}^2 \left(\mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0j}} \mathbf{I}_0^{*-1} \right)_{\eta\eta} (-i_0^{\eta\eta})^{-3/2} (\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0)_j \\
&+ \sum_{j=1}^2 \sum_{k^*=1}^2 \left\{ \left(-\frac{1}{4} \mathbf{I}_0^{*-1} \frac{\partial^2 \mathbf{I}_0^*}{\partial \theta_{0j} \partial \theta_{0k^*}} \mathbf{I}_0^{*-1} + \frac{1}{2} \mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0j}} \mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0k^*}} \mathbf{I}_0^{*-1} \right)_{\eta\eta} (-i_0^{\eta\eta})^{-3/2} \right. \\
&\quad \left. + \frac{3}{8} \left(\mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0j}} \mathbf{I}_0^{*-1} \right)_{\eta\eta} \left(\mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0k^*}} \mathbf{I}_0^{*-1} \right)_{\eta\eta} (-i_0^{\eta\eta})^{-5/2} \right\} \\
&\quad \times (\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0)_j (\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0)_{k^*} + O_p(n^{-3/2}) \\
&\equiv (-i_0^{\eta\eta})^{-1/2} + \mathbf{i}_\eta^{(1)'} (\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0) + \mathbf{i}_\eta^{(2)'} (\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0)^{<2>} + O_p(n^{-3/2}),
\end{aligned}$$

where

$$\begin{aligned}
\left(\mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{01}} \mathbf{I}_0^{*-1} \right)_{\eta\eta} &= c_2 \frac{-c_1(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2} c_2 = -c_1 c_2^2 \frac{1-2\theta_0}{\{\theta_0(1-\theta_0)\}^2}, \\
\left(\mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{02}} \mathbf{I}_0^{*-1} \right)_{\eta\eta} &= -c_1^2 c_2 \frac{1-2\theta_0}{\{\theta_0(1-\theta_0)\}^2},
\end{aligned}$$

consequently,

$$\begin{aligned}
(\mathbf{i}_\eta^{(1)})_j &= -\frac{1}{2} \left(\mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0j}} \mathbf{I}_0^{*-1} \right)_{\eta\eta} (-i_0^{\eta\eta})^{-3/2} \\
&= \frac{1}{2} \frac{(c_1 c_2)^2}{c_j} \frac{1-2\theta_0}{\{\theta_0(1-\theta_0)\}^2} \left\{ \frac{c_1 c_2}{\theta_0(1-\theta_0)} \right\}^{-3/2} \\
&= \frac{1}{2} \frac{(c_1 c_2)^{1/2}}{c_j} \frac{1-2\theta_0}{\{\theta_0(1-\theta_0)\}^{1/2}} \quad (j=1,2),
\end{aligned}$$

$$\text{where } \left(\mathbf{I}_0^{*-1} \frac{\partial \mathbf{I}_0^*}{\partial \theta_{0j}} \mathbf{I}_0^{*-1} \right)_{\eta\eta} = -\frac{(c_1 c_2)^2}{c_j} \frac{1-2\theta_0}{\{\theta_0(1-\theta_0)\}^2}.$$

$$\begin{aligned}
(\mathbf{i}_\eta^{(2)})_{(jk^*)} &= \left[-\frac{\delta_{jk^*}}{2} \frac{(c_1 c_2)^2}{c_j} \left(\frac{1}{\{\theta_0(1-\theta_0)\}^2} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} \right) \right. \\
&\quad \left. + \frac{1}{2} (-1)^{1-\delta_{jk^*}} (c_1 c_2)^2 \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} \right] \left\{ \frac{c_1 c_2}{\theta_0(1-\theta_0)} \right\}^{-3/2} \\
&\quad + \frac{3}{8} \frac{c_1^4 c_2^4}{c_j c_{k^*}} \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^4} \left\{ \frac{c_1 c_2}{\theta_0(1-\theta_0)} \right\}^{-5/2} \\
&= -\frac{\delta_{jk^*}}{2} \frac{(c_1 c_2)^{1/2}}{c_j} \left(\frac{1}{\{\theta_0(1-\theta_0)\}^{1/2}} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^{3/2}} \right) \\
&\quad + \frac{1}{2} (-1)^{1-\delta_{jk^*}} (c_1 c_2)^{1/2} (1-2\theta_0)^2 \{\theta_0(1-\theta_0)\}^{-3/2} \\
&\quad + \frac{3}{8} \frac{(c_1 c_2)^{3/2}}{c_j c_{k^*}} (1-2\theta_0)^2 \{\theta_0(1-\theta_0)\}^{-3/2}.
\end{aligned}$$

Note that while $t_{w\theta} \equiv n^{1/2} (\hat{i}_W^{\theta\theta})^{-1/2} (\hat{\theta}_W - \theta_0)$, $t_{w\eta} \equiv n^{-1/2} (-\hat{i}_W^{\eta\eta})^{-1/2} \hat{\eta}_W$.

Alternatively, $t_{w\eta} = n^{1/2} (-\hat{i}_W^{\eta\eta})^{-1/2} (n^{-1} \hat{\eta}_W)$, where

$$n^{-1} \hat{\eta}_W = \frac{n_W}{n} \frac{c_{W1} c_{W2} (\hat{p}_{W2} - \hat{p}_{W1})}{\hat{\theta}_W (1 - \hat{\theta}_W)} = O_p(n^{-1/2}).$$

4. Results using special properties of Example 2.1 (Example 1.2)

4.1 Partial derivatives of $\hat{\eta}$ with respect to \hat{p}_1 and \hat{p}_2

Note that $\hat{\theta} = c_1 \hat{p}_1 + c_2 \hat{p}_2$ and $n^{-1} \hat{\eta} = \frac{c_1 c_2 (\hat{p}_2 - \hat{p}_1)}{\hat{\theta} (1 - \hat{\theta})}$, then

$$\begin{aligned}
n^{-1} \frac{\partial \hat{\eta}}{\partial \hat{\mathbf{p}}} \Big|_{\hat{\mathbf{p}}=\theta_0} &= c_1 c_2 \left\{ \frac{(-1, 1)'}{\hat{\theta} (1 - \hat{\theta})} - \frac{\hat{p}_2 - \hat{p}_1}{\{\hat{\theta} (1 - \hat{\theta})\}^2} (1 - 2\hat{\theta}) (c_1, c_2)' \right\} \Big|_{\hat{\mathbf{p}}=\theta_0} \\
&= c_1 c_2 \frac{(-1, 1)'}{\theta_0 (1 - \theta_0)},
\end{aligned}$$

$$\begin{aligned}
n^{-1} \frac{\partial^2 \hat{\eta}}{(\partial \hat{\mathbf{p}})^{\langle 2 \rangle}} \Big|_{\hat{\mathbf{p}}=\boldsymbol{\theta}_0} &= c_1 c_2 \left\{ -\frac{\begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix}}{\{\hat{\theta}(1-\hat{\theta})\}^2} (1-2\hat{\theta}) \right. \\
&\quad \left. + 2 \frac{\hat{p}_2 - \hat{p}_1}{\{\hat{\theta}(1-\hat{\theta})\}^2} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^{\langle 2 \rangle} + \frac{2(\hat{p}_2 - \hat{p}_1)}{\{\hat{\theta}(1-\hat{\theta})\}^3} (1-2\hat{\theta})^2 \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^{\langle 2 \rangle} \right\} \Big|_{\hat{\mathbf{p}}=\boldsymbol{\theta}_0} \\
&= -\frac{c_1 c_2 (1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2} \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} \\
&= -\frac{c_1 c_2 (1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2} (-2c_1, c_1 - c_2, c_1 - c_2, 2c_2)', \\
n^{-1} \frac{\partial^3 \hat{\eta}}{(\partial \hat{\mathbf{p}})^{\langle 3 \rangle}} \Big|_{\hat{\mathbf{p}}=\boldsymbol{\theta}_0} &= 2c_1 c_2 \left[\frac{1}{\{\theta_0(1-\theta_0)\}^2} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} \right] \\
&\times \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^{\langle 2 \rangle} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^{\langle 2 \rangle} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} \\
&= 2c_1 c_2 \left[\frac{1}{\{\theta_0(1-\theta_0)\}^2} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} \right] \\
&\times \{(-2c_1^2, -2c_1 c_2, c_1^2 - c_1 c_2, c_1 c_2 - c_2^2, c_1^2 - c_1 c_2, c_1 c_2 - c_2^2, 2c_1 c_2, 2c_2^2)' \\
&\quad + (-c_1^2, c_1^2, -c_1 c_2, c_1 c_2, -c_1 c_2, c_1 c_2, -c_2^2, c_2^2)'\} \\
&= 2c_1 c_2 \left[\frac{1}{\{\theta_0(1-\theta_0)\}^2} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} \right] (-3c_1^2, c_1^2 - 2c_1 c_2, c_1^2 - 2c_1 c_2, \\
&\quad 2c_1 c_2 - c_2^2, c_1^2 - 2c_1 c_2, 2c_1 c_2 - c_2^2, 2c_1 c_2 - c_2^2, 3c_2^2)'
\end{aligned}$$

where

$$\begin{aligned}
\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^{\langle 2 \rangle} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} &= (c_1^2, c_1 c_2, c_1 c_2, c_2^2)' \otimes (-1, 1)' \\
&= (-c_1^2, c_1^2, -c_1 c_2, c_1 c_2, -c_1 c_2, c_1 c_2, -c_2^2, c_2^2)'.
\end{aligned}$$

4.2 Asymptotic cumulants of $\hat{\theta}$ and $\hat{\eta}$ by ML

Since $\hat{\theta} = c_1 \hat{p}_1 + c_2 \hat{p}_2$ is a usual sample proportion of size n ,

$$\kappa_1(\hat{\theta} - \theta_0) = 0,$$

$$\kappa_2(\hat{\theta}) = n^{-1} \theta_0 (1 - \theta_0),$$

$$\kappa_3(\hat{\theta}) = n^{-2} \theta_0 (1 - \theta_0) (1 - 2\theta_0),$$

$$\kappa_4(\hat{\theta}) = n^{-3} \theta_0 (1 - \theta_0) \{1 - 6\theta_0 (1 - \theta_0)\}.$$

On the other hand, since

$$n^{-1} \hat{\eta} = \frac{c_1 c_2 (\hat{p}_2 - \hat{p}_1)}{\hat{\theta} (1 - \hat{\theta})} = \frac{c_1 c_2 (\hat{p}_2 - \hat{p}_1)}{(c_1 \hat{p}_1 + c_2 \hat{p}_2) (1 - c_1 \hat{p}_1 - c_2 \hat{p}_2)},$$

we use

$$\kappa_1(\hat{\mathbf{p}}) = 0,$$

$$\kappa_2(\hat{\mathbf{p}}) = n^{-1} \theta_0 (1 - \theta_0) \text{vec diag}(c_1^{-1}, c_2^{-1}),$$

$$\kappa_3(\hat{\mathbf{p}}) = n^{-2} \theta_0 (1 - \theta_0) (1 - 2\theta_0) (c_1^{-2}, \mathbf{0}'_{(6)}, c_2^{-2})',$$

$$\kappa_4(\hat{\mathbf{p}}) = n^{-3} \theta_0 (1 - \theta_0) \{1 - 6\theta_0 (1 - \theta_0)\} (c_1^{-3}, \mathbf{0}'_{(14)}, c_2^{-3})'$$

$$(n \text{ cov}(\hat{\mathbf{p}}) = \theta_0 (1 - \theta_0) \text{diag}(c_1^{-1}, c_2^{-1})).$$

$$\begin{aligned} \kappa_1(n^{-1} \hat{\eta}) &= \frac{1}{2} \frac{n^{-1} \partial^2 \eta_0}{(\partial \mathbf{p}')^{<2>}} E_{\theta} \{(\hat{\mathbf{p}} - \mathbf{p})\}^{<2>} + O(n^{-2}) \\ &= \frac{1}{2} \frac{n^{-1} \partial^2 \eta_0}{(\partial \mathbf{p}')^{<2>}} n^{-1} \theta_0 (1 - \theta_0) \text{vec diag}(c_1^{-1}, c_2^{-1}) + O(n^{-2}) \\ &= -n^{-1} \frac{c_1 c_2 (1 - 2\theta_0)}{2\theta_0 (1 - \theta_0)} \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} \text{vec diag}(c_1^{-1}, c_2^{-1}) \\ &\quad + O(n^{-2}) \\ &= -n^{-1} \frac{c_1 c_2 (1 - 2\theta_0)}{\theta_0 (1 - \theta_0)} (-1 + 1) + O(n^{-2}) \\ &= O(n^{-2}) \quad (\alpha_{\eta_1} = 0). \end{aligned}$$

$$\begin{aligned}
\kappa_2(n^{-1}\hat{\eta}) &= \frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'} n^{-1}\theta_0(1-\theta_0) \text{diag}(c_1^{-1}, c_2^{-1}) \frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}} \\
&+ \frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'} \kappa_3(\hat{\mathbf{p}}, \hat{\mathbf{p}}'^{\langle 2 \rangle}) \frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p})^{\langle 2 \rangle}} + \frac{1}{4} \frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p}')^{\langle 2 \rangle}} \kappa_2(\hat{\mathbf{p}}^{\langle 2 \rangle}, \hat{\mathbf{p}}'^{\langle 2 \rangle}) \frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p})^{\langle 2 \rangle}} \\
&- n^{-2}\alpha_{\eta_1}^2 + \frac{1}{3} \frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'} \kappa_2(\hat{\mathbf{p}}, \hat{\mathbf{p}}'^{\langle 3 \rangle}) \frac{n^{-1}\partial^3\eta_0}{(\partial\mathbf{p})^{\langle 3 \rangle}} + O(n^{-3}) \\
&= n^{-1}c_1c_2 \frac{(-1, 1)}{\theta_0(1-\theta_0)} \theta_0(1-\theta_0) \text{diag}(c_1^{-1}, c_2^{-1}) c_1c_2 \frac{(-1, 1)'}{\theta_0(1-\theta_0)} \\
&- n^{-2} \left[c_1c_2 \frac{(-1, 1)}{\theta_0(1-\theta_0)} \otimes \left\{ \frac{c_1c_2(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^2} \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} \right\} \right] \\
&\quad \times \theta_0(1-\theta_0)(1-2\theta_0)(c_1^{-2}, \mathbf{0}'_{(6)}, c_2^{-2})' \\
&+ \frac{n^{-2}}{2} \frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p}')^{\langle 2 \rangle}} \{n \text{cov}(\hat{\mathbf{p}})\}^{\langle 2 \rangle} \frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p})^{\langle 2 \rangle}} \\
&+ n^{-2} \frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'} [n \text{cov}(\hat{\mathbf{p}}) \otimes \text{vec}'\{n \text{cov}(\hat{\mathbf{p}})\}] \frac{n^{-1}\partial^3\eta_0}{(\partial\mathbf{p})^{\langle 3 \rangle}} + O(n^{-3}) \\
&= n^{-1} \frac{c_1c_2}{\theta_0(1-\theta_0)} - n^{-2} 2(c_1c_2)^2 \frac{1}{c_1c_2} \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2} \\
&+ \frac{n^{-2}}{2} \frac{(c_1c_2)^2(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^4} \{\theta_0(1-\theta_0)\}^2 (-2c_1, c_1 - c_2, c_1 - c_2, 2c_2) \\
&\quad \times \text{diag}\left(\frac{1}{c_1^2}, \frac{1}{c_1c_2}, \frac{1}{c_1c_2}, \frac{1}{c_2^2}\right) (-2c_1, c_1 - c_2, c_1 - c_2, 2c_2)' \\
&+ n^{-2} 2 \frac{(c_1c_2)^2}{\theta_0(1-\theta_0)} (-1, 1) \left\{ \begin{pmatrix} c_1^{-1} \mathbf{0} \\ 0 & c_2^{-1} \end{pmatrix} \otimes (c_1^{-1}, 0, 0, c_2^{-1}) \right\} \\
&\times \left[\frac{1}{\{\theta_0(1-\theta_0)\}^2} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} \right] \{\theta_0(1-\theta_0)\}^2 \\
&\times (-3c_1^2, c_1^2 - 2c_1c_2, c_1^2 - 2c_1c_2, 2c_1c_2 - c_2^2, c_1^2 - 2c_1c_2, 2c_1c_2 - c_2^2, \\
&\quad 2c_1c_2 - c_2^2, 3c_2^2)' + O(n^{-3})
\end{aligned}$$

$$\begin{aligned}
&= n^{-1} \frac{c_1 c_2}{\theta_0(1-\theta_0)} - n^{-2} 2c_1 c_2 \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2} \\
&\quad + \frac{n^{-2} (c_1 c_2)^2 (1-2\theta_0)^2}{2 \{\theta_0(1-\theta_0)\}^2} \left\{ 4 + \frac{2(c_1 - c_2)^2}{c_1 c_2} + 4 \right\} \\
&\quad + n^{-2} 2(c_1 c_2)^2 \left[\frac{1}{\theta_0(1-\theta_0)} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2} \right] \\
&\quad \quad \times (-1, 1)(-3 + 2 - c_1^{-1} c_2, c_1 c_2^{-1} - 2 + 3)' + O(n^{-3}) \\
&= n^{-1} \frac{c_1 c_2}{\theta_0(1-\theta_0)} - n^{-2} 2c_1 c_2 \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2} \\
&\quad + n^{-2} \{4(c_1 c_2)^2 + c_1 c_2 (c_1 - c_2)^2\} \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2} \\
&\quad + n^{-2} 2(c_1 c_2)^2 \left[\frac{1}{\theta_0(1-\theta_0)} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^2} \right] (2 + c_1^{-1} c_2 + c_1 c_2^{-1}) + O(n^{-3}) \\
&= n^{-1} \frac{c_1 c_2}{\theta_0(1-\theta_0)} + n^{-2} \frac{c_1 c_2}{\theta_0(1-\theta_0)} \left\{ 2 + \frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)} \right\} + O(n^{-3}) \\
&\equiv n^{-1} \alpha_{\eta_2} + n^{-2} \alpha_{\eta_{\Delta 2}} + O(n^{-3}),
\end{aligned}$$

where $4(c_1 c_2)^2 + c_1 c_2 (c_1 - c_2)^2 = c_1 c_2$ and $2 + c_1^{-1} c_2 + c_1 c_2^{-1} = 1 / (c_1 c_2)$ are used.

$$\begin{aligned}
\kappa_3(n^{-1} \hat{\eta}) &= n^{-2} \left(\frac{n^{-1} \partial \eta_0}{\partial \mathbf{p}'} \right)^{\langle 3 \rangle} n^2 \kappa_3(\hat{\mathbf{p}}) \\
&\quad + n^{-2} \frac{3}{2} \left\{ \frac{n^{-1} \partial^2 \eta_0}{(\partial \mathbf{p}')^{\langle 2 \rangle}} \otimes \left(\frac{n^{-1} \partial \eta_0}{\partial \mathbf{p}'} \right)^{\langle 2 \rangle} \right\} \sum^{(3)} \{\text{vec } n \text{ cov}(\hat{\mathbf{p}})\}^{\langle 2 \rangle} \\
&\quad - n^{-2} \frac{3}{2} \frac{n^{-1} \partial^2 \eta_0}{(\partial \mathbf{p}')^{\langle 2 \rangle}} \text{vec } n \text{ cov}(\hat{\mathbf{p}}) \alpha_{\eta_2} + O(n^{-3})
\end{aligned}$$

$$\begin{aligned}
&= n^{-2} \frac{(c_1 c_2)^3}{\{\theta_0(1-\theta_0)\}^3} (-1, 1)^{\langle 3 \rangle} \theta_0(1-\theta_0)(1-2\theta_0)(c_1^{-2}, \mathbf{0}'_{(6)}, c_2^{-2})' \\
&+ n^{-2} 3 \frac{n^{-1} \partial \eta_0}{\partial \mathbf{p}'} n \text{cov}(\hat{\mathbf{p}}) \frac{n^{-1} \partial^2 \eta_0}{\partial \mathbf{p} \partial \mathbf{p}'} n \text{cov}(\hat{\mathbf{p}}) \frac{n^{-1} \partial \eta_0}{\partial \mathbf{p}} + O(n^{-3}) \\
&= n^{-2} \frac{c_1 c_2 (c_1 - c_2)}{\{\theta_0(1-\theta_0)\}^2} (1-2\theta_0) - n^{-2} 3 \frac{(c_1 c_2)^3}{\{\theta_0(1-\theta_0)\}^2} (1-2\theta_0) \\
&\times (-1, 1) \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix} \begin{pmatrix} -2c_1 & c_1 - c_2 \\ c_1 - c_2 & 2c_2 \end{pmatrix} \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + O(n^{-3}) \\
&= n^{-2} \frac{c_1 c_2 (c_1 - c_2)}{\{\theta_0(1-\theta_0)\}^2} (1-2\theta_0) - n^{-2} 3 \frac{(c_1 c_2)^3}{\{\theta_0(1-\theta_0)\}^2} (1-2\theta_0) \\
&\quad \times (-c_1^{-1}, c_2^{-1}) \begin{pmatrix} -2c_1 & c_1 - c_2 \\ c_1 - c_2 & 2c_2 \end{pmatrix} \begin{pmatrix} -c_1^{-1} \\ c_2^{-1} \end{pmatrix} + O(n^{-3}) \\
&= n^{-2} \frac{c_1 c_2 (c_1 - c_2)}{\{\theta_0(1-\theta_0)\}^2} (1-2\theta_0) - n^{-2} 3 \frac{(c_1 c_2)^3}{\{\theta_0(1-\theta_0)\}^2} (1-2\theta_0) \\
&\quad \times \left\{ -2c_1^{-1} - 2 \frac{c_1 - c_2}{c_1 c_2} + 2c_2^{-1} \right\} + O(n^{-3}) \\
&= n^{-2} c_1 c_2 (c_1 - c_2) \frac{1-2\theta_0}{\{\theta_0(1-\theta_0)\}^2} + O(n^{-3}) \\
&\equiv n^{-2} \beta_{\eta_3} + O(n^{-3}),
\end{aligned}$$

$$\begin{aligned}
\kappa_4(n^{-1}\hat{\eta}) &= n^{-3} \left(\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'} \right)^{\langle 4 \rangle} n^3 \kappa_4(\hat{\mathbf{p}}) \\
&+ 2n^{-3} \left\{ \left(\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'} \right)^{\langle 3 \rangle} \otimes \frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p}')^{\langle 2 \rangle}} \right\} \sum^{(10)} \{ \text{vec } n \text{ cov}(\hat{\mathbf{p}}) \otimes n^2 \kappa_3(\hat{\mathbf{p}}) \} \\
&+ n^{-3} \left\{ \frac{3}{2} \left(\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'} \right)^{\langle 2 \rangle} \otimes \left(\frac{n^{-1}\partial^2\eta_0}{(\partial\mathbf{p}')^{\langle 2 \rangle}} \right)^{\langle 2 \rangle} + \frac{2}{3} \left(\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'} \right)^{\langle 3 \rangle} \otimes \frac{n^{-1}\partial^3\eta_0}{(\partial\mathbf{p}')^{\langle 3 \rangle}} \right\} \\
&\times \sum^{(15)} \{ \text{vec } n \text{ cov}(\hat{\mathbf{p}}) \}^{\langle 3 \rangle} - 6n^{-3} \alpha_{\eta_2} \alpha_{\eta_{\Delta 2}} + O(n^{-4}),
\end{aligned}$$

where the first term is

$$\begin{aligned}
&n^{-3} \left(\frac{n^{-1}\partial\eta_0}{\partial\mathbf{p}'} \right)^{\langle 4 \rangle} n^3 \kappa_4(\hat{\mathbf{p}}) \\
&= n^{-3} \frac{(c_1 c_2)^4}{\{\theta_0(1-\theta_0)\}^4} (-1, 1)^{\langle 4 \rangle} \theta_0(1-\theta_0) \{1-6\theta_0(1-\theta_0)\} (c_1^{-3}, \mathbf{0}'_{(14)}, c_2^{-3})' \\
&= n^{-3} (c_1 c_2^4 + c_1^4 c_2) \frac{1-6\theta_0(1-\theta_0)}{\{\theta_0(1-\theta_0)\}^3} \\
&= n^{-3} c_1 c_2 (1-3c_1 c_2) \frac{1}{\{\theta_0(1-\theta_0)\}^2} \left\{ \frac{1}{\theta_0(1-\theta_0)} - 6 \right\},
\end{aligned}$$

the second term is

$$\begin{aligned}
& 2n^{-3} \left\{ \left(\frac{n^{-1} \partial \eta_0}{\partial \mathbf{p}'} \right)^{\langle 3 \rangle} \otimes \frac{n^{-1} \partial^2 \eta_0}{(\partial \mathbf{p}')^{\langle 2 \rangle}} \right\} \sum^{(10)} \{ \text{vec } n \text{ cov}(\hat{\mathbf{p}}) \otimes n^2 \kappa_3(\hat{\mathbf{p}}) \} \\
&= -n^{-3} 2 \frac{(c_1 c_2)^4}{\{\theta_0(1-\theta_0)\}^5} (1-2\theta_0) \{(-1,1)^{\langle 3 \rangle} \otimes (-2c_1, c_1 - c_2, c_1 - c_2, 2c_2)\} \\
&\quad \times \{\theta_0(1-\theta_0)\}^2 (1-2\theta_0) \sum^{(10)} (c_1^{-1}, 0, 0, c_2^{-1})' \otimes (c_1^{-2}, \mathbf{0}'_{(6)}, c_2^{-2})' \\
&= -n^{-3} 2 \frac{(c_1 c_2)^4}{\{\theta_0(1-\theta_0)\}^3} (1-2\theta_0)^2 \{(-1,1)^{\langle 3 \rangle} \otimes (-2c_1, c_1 - c_2, c_1 - c_2, 2c_2)\} \\
&\quad \times \sum^{(10)} (c_1^{-1}, 0, 0, c_2^{-1})' \otimes (c_1^{-2}, \mathbf{0}'_{(6)}, c_2^{-2})'
\end{aligned}$$

and the third term is

$$\begin{aligned}
& n^{-3} \left\{ \frac{3}{2} \left(\frac{n^{-1} \partial \eta_0}{\partial \mathbf{p}'} \right)^{\langle 2 \rangle} \otimes \left(\frac{n^{-1} \partial^2 \eta_0}{(\partial \mathbf{p}')^{\langle 2 \rangle}} \right)^{\langle 2 \rangle} + \frac{2}{3} \left(\frac{n^{-1} \partial \eta_0}{\partial \mathbf{p}'} \right)^{\langle 3 \rangle} \otimes \frac{n^{-1} \partial^3 \eta_0}{(\partial \mathbf{p}')^{\langle 3 \rangle}} \right\} \\
&\quad \times \sum^{(15)} \{ \text{vec } n \text{ cov}(\hat{\mathbf{p}}) \}^{\langle 3 \rangle} \\
&= n^{-3} \left[\frac{3}{2} \frac{(c_1 c_2)^4}{\{\theta_0(1-\theta_0)\}^6} (1-2\theta_0)^2 \{(-1,1)^{\langle 2 \rangle} \otimes (-2c_1, c_1 - c_2, c_1 - c_2, 2c_2)^{\langle 2 \rangle}\} \right. \\
&\quad + \frac{2}{3} \frac{(c_1 c_2)^4}{\{\theta_0(1-\theta_0)\}^3} 2 \left[\frac{1}{\{\theta_0(1-\theta_0)\}^2} + \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^3} \right] \\
&\quad \times \{(-1,1)^{\langle 3 \rangle} \otimes (-3c_1^2, c_1^2 - 2c_1 c_2, c_1^2 - 2c_1 c_2, 2c_1 c_2 - c_2^2, c_1^2 - 2c_1 c_2, \\
&\quad \left. 2c_1 c_2 - c_2^2, 2c_1 c_2 - c_2^2, 3c_2^2)\}' \} \{\theta_0(1-\theta_0)\}^3 \sum^{(15)} (c_1^{-1}, 0, 0, c_2^{-1})^{\langle 3 \rangle} \right].
\end{aligned}$$

Then,

$$\begin{aligned}
\kappa_4(n^{-1}\hat{\eta}) &= n^{-3} \left[\frac{c_1 c_2 (1 - 3c_1 c_2)}{\{\theta_0(1 - \theta_0)\}^2} \left\{ \frac{1}{\theta_0(1 - \theta_0)} - 6 \right\} \right. \\
&\quad - 2 \frac{(c_1 c_2)^4}{\{\theta_0(1 - \theta_0)\}^3} (1 - 2\theta_0)^2 \{(-1, 1)^{\langle 3 \rangle} \otimes (-2c_1, c_1 - c_2, c_1 - c_2, 2c_2)\} \\
&\quad \times \sum_{(10)} (c_1^{-1}, 0, 0, c_2^{-1})' \otimes (c_1^{-2}, \mathbf{0}'_{(6)}, c_2^{-2})' \\
&\quad + (c_1 c_2)^4 \left[\frac{3}{2} \frac{(1 - 2\theta_0)^2}{\{\theta_0(1 - \theta_0)\}^3} \{(-1, 1)^{\langle 2 \rangle} \otimes (-2c_1, c_1 - c_2, c_1 - c_2, 2c_2)^{\langle 2 \rangle}\} \right. \\
&\quad \left. + \frac{4}{3} \left[\frac{1}{\{\theta_0(1 - \theta_0)\}^2} + \frac{(1 - 2\theta_0)^2}{\{\theta_0(1 - \theta_0)\}^3} \right] \right. \\
&\quad \times \{(-1, 1)^{\langle 3 \rangle} \otimes (-3c_1^2, c_1^2 - 2c_1 c_2, c_1^2 - 2c_1 c_2, 2c_1 c_2 - c_2^2, c_1^2 - 2c_1 c_2, \\
&\quad \left. 2c_1 c_2 - c_2^2, 2c_1 c_2 - c_2^2, 3c_2^2)\}' \} \left. \sum_{(15)} (c_1^{-1}, 0, 0, c_2^{-1})^{\langle 3 \rangle} - 6\alpha_{\eta_2} \alpha_{\eta_{\Delta 2}} \right] + O(n^{-4}) \right] \\
&\quad \left. \right]_{(A)} \\
&\equiv n^{-3} \alpha_{\eta_4} + O(n^{-4}),
\end{aligned}$$

4.3 Asymptotic cumulants of t_θ and t_η by maximum likelihood

$t_\theta = \frac{n^{1/2}(\hat{\theta} - \theta_0)}{\{\hat{\theta}(1 - \hat{\theta})\}^{1/2}}$, where $\hat{\theta}$ is the MLE. The asymptotic cumulants of t_θ

reduce to those of the studentized sample proportion (see e.g., Ogasawara, 2012, p.12):

$$p \equiv \theta_0, \quad q \equiv 1 - \theta_0,$$

$$\kappa_1(t_\theta) = -n^{-1/2} \frac{(pq)^{-1/2}}{2} (1 - 2p) + O(n^{-3/2}) \equiv n^{-1/2} \alpha_{\theta_1}^{(t)} + O(n^{-2/3}),$$

$$\kappa_2(t_\theta) = 1 + n^{-1} \left\{ \frac{7}{4} (1 - 2p)^2 (pq)^{-1} + 3 \right\} + O(n^{-2}) \equiv 1 + n^{-1} \alpha_{\theta_{\Delta 2}}^{(t)} + O(n^{-2})$$

$$(\alpha_{\theta_2}^{(t)} = 1),$$

$$\kappa_3(t_\theta) = -n^{-1/2} 2(pq)^{-1/2} (1 - 2p) + O(n^{-3/2}) \equiv n^{-1/2} \alpha_{\theta_3}^{(t)} + O(n^{-2/3}),$$

$$\kappa_4(t_\theta) = n^{-1} \{ (pq)^{-1} + 9(1 - 2p)^2 (pq)^{-1} + 6 \} + O(n^{-3/2}) \equiv n^{-1} \alpha_{\theta_4}^{(t)} + O(n^{-2}),$$

$$n \operatorname{acov}(\hat{\theta}, \hat{\alpha}_{\theta_1}^{(t)}) = \frac{1}{4} (1 - 2p)^2 (pq)^{-1/2} + (pq)^{1/2},$$

$$n \operatorname{acov}(\hat{\theta}, \hat{\alpha}_{\theta_3}^{(t)}) = (1 - 2p)^2 (pq)^{-1/2} + 4(pq)^{1/2}.$$

$$\begin{aligned} t_\eta &= \frac{n^{1/2} (n^{-1} \hat{\eta})}{(-\hat{i}^{\eta\eta})^{1/2}} = \frac{n^{1/2} c_1 c_2 (\hat{p}_2 - \hat{p}_1) / \{\hat{\theta}(1 - \hat{\theta})\}}{[c_1 c_2 / \{\hat{\theta}(1 - \hat{\theta})\}]^{1/2}} \\ &= \frac{n^{1/2} (c_1 c_2)^{1/2} (\hat{p}_2 - \hat{p}_1)}{\{\hat{\theta}(1 - \hat{\theta})\}^{1/2}} = \frac{\hat{p}_2 - \hat{p}_1}{\{\hat{\theta}(1 - \hat{\theta}) / (n c_1 c_2)\}^{1/2}}, \end{aligned}$$

where $\hat{\eta} = n \frac{c_1 c_2 (\hat{p}_2 - \hat{p}_1)}{\hat{\theta}(1 - \hat{\theta})}$ is the MLE and $\hat{i}^{\eta\eta} = -\frac{c_1 c_2}{\hat{\theta}(1 - \hat{\theta})}$.

Noting that $\frac{\hat{\theta}(1 - \hat{\theta})}{n c_1 c_2} = \left(\frac{1}{n_1} + \frac{1}{n_2} \right) \hat{\theta}(1 - \hat{\theta})$ and

$\operatorname{var}(\hat{p}_1 - \hat{p}_2) = \left(\frac{1}{n_1} + \frac{1}{n_2} \right) \theta_0 (1 - \theta_0)$, we find that t_η is the studentized

$\hat{p}_2 - \hat{p}_1$.

$$\frac{\partial(\hat{p}_2 - \hat{p}_1)}{\partial \hat{\mathbf{p}}} = (-1, 1)', \quad \frac{\partial \hat{\theta}}{\partial \hat{\mathbf{p}}} = \frac{\partial(c_1 \hat{p}_1 + c_2 \hat{p}_2)}{\partial \hat{\mathbf{p}}} = (c_1, c_2)',$$

$$\frac{\partial t_\eta}{\partial \hat{\mathbf{p}}} = n^{1/2} (c_1 c_2)^{1/2} \left[\frac{(-1, 1)'}{\{\hat{\theta}(1 - \hat{\theta})\}^{1/2}} - \frac{\hat{p}_2 - \hat{p}_1}{2\{\hat{\theta}(1 - \hat{\theta})\}^{3/2}} (1 - 2\hat{\theta})(c_1, c_2)' \right],$$

$$\begin{aligned} \frac{\partial^2 t_\eta}{(\partial \hat{\mathbf{p}})^{\langle 2 \rangle}} &= n^{1/2} (c_1 c_2)^{1/2} \left[-\frac{1 - 2\hat{\theta}}{2\{\hat{\theta}(1 - \hat{\theta})\}^{3/2}} \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} \right. \\ &\quad \left. + (\hat{p}_2 - \hat{p}_1) \left[\frac{1}{\{\hat{\theta}(1 - \hat{\theta})\}^{3/2}} + \frac{3}{4} \frac{(1 - 2\hat{\theta})^2}{\{\hat{\theta}(1 - \hat{\theta})\}^{5/2}} \right] \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^{\langle 2 \rangle} \right], \end{aligned}$$

$$\begin{aligned} \frac{\partial^3 t_\eta}{(\partial \hat{\mathbf{p}})^{\langle 3 \rangle}} &= n^{1/2} (c_1 c_2)^{1/2} \left[\frac{1}{\{\theta_0(1 - \theta_0)\}^{3/2}} + \frac{3}{4} \frac{(1 - 2\theta_0)^2}{\{\theta_0(1 - \theta_0)\}^{5/2}} \right] \\ &\quad \times \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^{\langle 2 \rangle} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^{\langle 2 \rangle} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

$$\begin{aligned} \kappa_1(t_\eta) &= \frac{1}{2} \frac{\partial^2 t_\eta}{(\partial \hat{\mathbf{p}}')^{\langle 2 \rangle}} \mathbb{E}_\theta \{ (\hat{\mathbf{p}} - \mathbf{p})^{\langle 2 \rangle} \} + O(n^{-3/2}) \\ &= \frac{1}{2} \frac{\partial^2 t_\eta}{(\partial \hat{\mathbf{p}}')^{\langle 2 \rangle}} n^{-1} \theta_0 (1 - \theta_0) \text{vec diag}(c_1^{-1}, c_2^{-1}) + O(n^{-3/2}) \\ &= -\frac{n^{-1/2} (c_1 c_2)^{1/2} (1 - 2\theta_0)}{4\{\theta_0(1 - \theta_0)\}^{1/2}} \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}' \text{vec diag}(c_1^{-1}, c_2^{-1}) \\ &\quad + O(n^{-3/2}) \\ &= -\frac{n^{-1/2} (c_1 c_2)^{1/2} (1 - 2\theta_0)}{2\{\theta_0(1 - \theta_0)\}^{1/2}} (-1 + 1) + O(n^{-3/2}) \\ &= O(n^{-3/2}) \quad (\alpha_{\eta 1}^{(t)} = 0), \end{aligned}$$

$$\begin{aligned}
\kappa_2(t_\eta) &= \frac{\partial t_\eta}{\partial \mathbf{p}'} n^{-1} \theta_0 (1 - \theta_0) \text{diag}(c_1^{-1}, c_2^{-1}) \frac{\partial t_\eta}{\partial \mathbf{p}} \\
&+ \frac{\partial t_\eta}{\partial \mathbf{p}'} \kappa_3(\hat{\mathbf{p}}, \hat{\mathbf{p}}'^{\langle 2 \rangle}) \frac{\partial^2 t_\eta}{(\partial \mathbf{p})^{\langle 2 \rangle}} + \frac{1}{4} \frac{\partial^2 t_\eta}{(\partial \mathbf{p}')^{\langle 2 \rangle}} \kappa_2(\hat{\mathbf{p}}^{\langle 2 \rangle}, \hat{\mathbf{p}}'^{\langle 2 \rangle}) \frac{\partial^2 t_\eta}{(\partial \mathbf{p})^{\langle 2 \rangle}} \\
&+ \frac{1}{3} \frac{\partial t_\eta}{\partial \mathbf{p}'} \kappa_2(\hat{\mathbf{p}}, \hat{\mathbf{p}}'^{\langle 3 \rangle}) \frac{\partial^3 t_\eta}{(\partial \mathbf{p})^{\langle 3 \rangle}} - n^{-1} (\alpha_{\eta 1}^{(t)})^2 + O(n^{-2}),
\end{aligned}$$

where the first term is

$$\frac{c_1 c_2}{\theta_0 (1 - \theta_0)} \theta_0 (1 - \theta_0) (-1, 1) \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1,$$

the second term is

$$\begin{aligned}
&-n^{-1} \frac{c_1 c_2 (1 - 2\theta_0)}{2\{\theta_0 (1 - \theta_0)\}^2} \theta_0 (1 - \theta_0) (1 - 2\theta_0) \\
&\quad \times \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} \right]' (c_1^{-2}, \mathbf{0}'_{(6)}, c_2^{-2})' \\
&= -n^{-1} \frac{c_1 c_2 (1 - 2\theta_0)^2}{\theta_0 (1 - \theta_0)} (c_1^{-1} + c_2^{-1}) = -n^{-1} \frac{(1 - 2\theta_0)^2}{\theta_0 (1 - \theta_0)},
\end{aligned}$$

the third term is

$$\begin{aligned}
&n^{-1} \frac{1}{2} \frac{\partial^2 t_\eta}{(\partial \mathbf{p}')^{\langle 2 \rangle}} \{n \text{cov}(\hat{\mathbf{p}})\}^{\langle 2 \rangle} \frac{\partial^2 t_\eta}{(\partial \mathbf{p})^{\langle 2 \rangle}} \\
&= n^{-1} \frac{c_1 c_2 (1 - 2\theta_0)^2}{8\{\theta_0 (1 - \theta_0)\}^3} \{\theta_0 (1 - \theta_0)\}^2 (-2c_1, c_1 - c_2, c_1 - c_2, 2c_2) \\
&\quad \times \text{diag}(c_1^{-2}, c_1^{-1} c_2^{-1}, c_2^{-1} c_1^{-1}, c_2^{-2}) (-2c_1, c_1 - c_2, c_1 - c_2, 2c_2)' \\
&= n^{-1} \frac{c_1 c_2 (1 - 2\theta_0)^2}{8\theta_0 (1 - \theta_0)} \left(4 + 2 \frac{(c_1 - c_2)^2}{c_1 c_2} + 4 \right) \\
&= n^{-1} \frac{(1 - 2\theta_0)^2}{4\theta_0 (1 - \theta_0)}
\end{aligned}$$

and the fourth term is

$$\begin{aligned}
& n^{-1}c_1c_2 \frac{1}{\{\theta_0(1-\theta_0)\}^{1/2}} \left[\frac{1}{\{\theta_0(1-\theta_0)\}^{3/2}} + \frac{3}{4} \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^{5/2}} \right] \\
& \times (-1,1)[n \text{cov}(\hat{\mathbf{p}}) \otimes \text{vec}'\{n \text{cov}(\hat{\mathbf{p}})\}] \\
& \times \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^{<2>} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^{<2>} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} \\
& = n^{-1}c_1c_2 \left\{ 1 + \frac{3}{4} \frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)} \right\} (-1,1) \left\{ \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix} \otimes (c_1^{-1}, 0, 0, c_2^{-1}) \right\} \\
& \quad \times (-3c_1^2, c_1^2 - 2c_1c_2, c_1^2 - 2c_1c_2, 2c_1c_2 - c_2^2, c_1^2 - 2c_1c_2, 2c_1c_2 - c_2^2, \\
& \quad 2c_1c_2 - c_2^2, 3c_2^2)' \\
& = n^{-1}c_1c_2 \left\{ 1 + \frac{3}{4} \frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)} \right\} (-1,1)(-3 + 2 - c_1^{-1}c_2, c_1c_2^{-1} - 2 + 3)' \\
& = n^{-1}c_1c_2 \left\{ 1 + \frac{3}{4} \frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)} \right\} (2 + c_1^{-1}c_2 + c_1c_2^{-1}) \\
& = n^{-1} \left\{ 1 + \frac{3}{4} \frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)} \right\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\kappa_2(t_\eta) &= 1 + n^{-1} \left\{ 1 + \left(-1 + \frac{1}{4} + \frac{3}{4} \right) \frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)} \right\} + O(n^{-2}) \\
&= 1 + n^{-1} + O(n^{-2}) \\
&= 1 + n^{-1} \alpha_{\eta\Delta 2}^{(t)} + O(n^{-2}) \quad (\alpha_{\eta 2}^{(t)} = 1). \\
\kappa_3(t_\eta) &= \left(\frac{\partial t_\eta}{\partial \mathbf{p}'} \right)^{<3>} \kappa_3(\hat{\mathbf{p}}) + \frac{3}{2} \left\{ \frac{\partial^2 t_\eta}{(\partial \mathbf{p}')^{<2>}} \otimes \left(\frac{\partial t_\eta}{\partial \mathbf{p}'} \right)^{<2>} \right\} \sum^{(3)} \{\text{vec cov}(\hat{\mathbf{p}})\}^{<2>} \\
& \quad - n^{-1/2} 3\alpha_{\eta 1}^{(t)} + O(n^{-3/2}),
\end{aligned}$$

where the first term is

$$\begin{aligned}
& n^{-1/2} \frac{(c_1 c_2)^{3/2}}{\{\theta_0(1-\theta_0)\}^{3/2}} \theta_0(1-\theta_0)(1-2\theta_0)(-1,1)^{\langle 3 \rangle} (c_1^{-2}, \mathbf{0}'_{(6)}, c_2^{-2})' \\
&= n^{-1/2} \frac{(c_1 c_2)^{3/2}(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^{1/2}} (-c_1^{-2} + c_2^{-2}) \\
&= n^{-1/2} \frac{(c_1 - c_2)(1-2\theta_0)}{\{c_1 c_2 \theta_0(1-\theta_0)\}^{1/2}}
\end{aligned}$$

the sum of the second and third (actually 0) terms is

$$\begin{aligned}
& 3 \frac{\partial t_\eta}{\partial \mathbf{p}'} \text{cov}(\hat{\mathbf{p}}) \frac{\partial^2 t_\eta}{\partial \mathbf{p} \partial \mathbf{p}'} \text{cov}(\hat{\mathbf{p}}) \frac{\partial t_\eta}{\partial \mathbf{p}} \\
&= -3n^{-1/2} \frac{(c_1 c_2)^{3/2}}{\theta_0(1-\theta_0)} \{\theta_0(1-\theta_0)\}^2 \frac{1-2\theta_0}{2\{\theta_0(1-\theta_0)\}^{3/2}} \\
&\quad \times (-1,1) \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix} \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} (c_1, c_2) + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} (-1,1) \right\} \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
&= -n^{-1/2} \frac{3}{2} \frac{(c_1 c_2)^{3/2}(1-2\theta_0)}{\{\theta_0(1-\theta_0)\}^{1/2}} (-c_1^{-1}, c_2^{-1}) \begin{pmatrix} -2c_1 & c_1 - c_2 \\ c_1 - c_2 & 2c_2 \end{pmatrix} \begin{pmatrix} -c_1^{-1} \\ c_2^{-1} \end{pmatrix} \\
&= 0,
\end{aligned}$$

where the quadratic form on the left-hand side of the last equation is 0.

Then,

$$\begin{aligned}
\kappa_3(t_\eta) &= n^{-1/2} \frac{(c_1 - c_2)(1-2\theta_0)}{\{c_1 c_2 \theta_0(1-\theta_0)\}^{1/2}} + O(n^{-3/2}) \\
&\equiv n^{-1/2} \alpha_{\eta 3}^{(t)} + O(n^{-3/2}).
\end{aligned}$$

$$\begin{aligned}
\kappa_4(t_\eta) &= \left(\frac{\partial t_\eta}{\partial \mathbf{p}'} \right)^{\langle 4 \rangle} \kappa_4(\hat{\mathbf{p}}) + 2 \left\{ \left(\frac{\partial t_\eta}{\partial \mathbf{p}'} \right)^{\langle 3 \rangle} \otimes \frac{\partial^2 t_\eta}{(\partial \mathbf{p}')^{\langle 2 \rangle}} \right\} \\
&\quad \times \sum^{(10)} \{ \text{vec cov}(\hat{\mathbf{p}}) \otimes \kappa_3(\hat{\mathbf{p}}) \} \\
&\quad + \left\{ \frac{3}{2} \left(\frac{\partial t_\eta}{\partial \mathbf{p}'} \right)^{\langle 2 \rangle} \otimes \left(\frac{\partial^2 t_\eta}{(\partial \mathbf{p}')^{\langle 2 \rangle}} \right)^{\langle 2 \rangle} + \frac{2}{3} \left(\frac{\partial t_\eta}{\partial \mathbf{p}'} \right)^{\langle 3 \rangle} \otimes \frac{\partial^3 t_\eta}{(\partial \mathbf{p}')^{\langle 3 \rangle}} \right\} \\
&\quad \times \sum^{(15)} \{ \text{vec cov}(\hat{\mathbf{p}}) \}^{\langle 3 \rangle} - n^{-1} 6\alpha_{\eta\Delta 2}^{(t)} + O(n^{-3}),
\end{aligned}$$

where the first term is

$$\begin{aligned}
&n^{-1} \frac{(c_1 c_2)^2}{\{\theta_0(1-\theta_0)\}^2} \theta_0(1-\theta_0) \{1-6\theta_0(1-\theta_0)\} (-1, 1)^{\langle 4 \rangle} (c_1^{-3}, \mathbf{0}'_{(14)}, c_2^{-3})' \\
&= n^{-1} \left(\frac{c_2^2}{c_1} + \frac{c_1^2}{c_2} \right) \left\{ \frac{1}{\theta_0(1-\theta_0)} - 6 \right\}.
\end{aligned}$$

Since $\frac{c_2^2}{c_1} + \frac{c_1^2}{c_2} = \frac{c_1^3 + c_2^3}{c_1 c_2} = \frac{1-3c_1 c_2(c_1 + c_2)}{c_1 c_2} = \frac{1}{c_1 c_2} - 3$, the first term becomes

$$n^{-1} \left(\frac{1}{c_1 c_2} - 3 \right) \left\{ \frac{1}{\theta_0(1-\theta_0)} - 6 \right\}.$$

The second term is

$$\begin{aligned}
&-n^{-1} 2 \frac{(c_1 c_2)^2}{\{\theta_0(1-\theta_0)\}^{3/2}} \frac{1-2\theta_0}{2\{\theta_0(1-\theta_0)\}^{3/2}} \{\theta_0(1-\theta_0)\}^2 (1-2\theta_0) \\
&\times \{(-1, 1)^{\langle 3 \rangle} \otimes (-2c_1, c_1 - c_2, c_1 - c_2, 2c_2)\} \sum^{(10)} (c_1^{-1}, 0, 0, c_2^{-1})' \otimes (c_1^{-2}, \mathbf{0}'_{(6)}, c_2^{-2})'
\end{aligned}$$

and the third term is

$$\begin{aligned}
& n^{-1} \left[\frac{3}{2} \frac{(c_1 c_2)^2}{\{\theta_0(1-\theta_0)\}^4} \frac{(1-2\theta_0)^2}{4} \{(-1,1)^{\langle 2 \rangle} \otimes (-2c_1, c_1 - c_2, c_1 - c_2, 2c_2)^{\langle 2 \rangle}\} \right. \\
& + \frac{2}{3} \frac{(c_1 c_2)^2}{\{\theta_0(1-\theta_0)\}^{3/2}} \left[\frac{1}{\{\theta_0(1-\theta_0)\}^{3/2}} + \frac{3}{4} \frac{(1-2\theta_0)^2}{\{\theta_0(1-\theta_0)\}^{5/2}} \right] \\
& \times (-1,1)^{\langle 3 \rangle} \otimes (-3c_1^2, c_1^2 - 2c_1 c_2, c_1^2 - 2c_1 c_2, 2c_1 c_2 - c_2^2, c_1^2 - 2c_1 c_2, 2c_1 c_2 - c_2^2, \\
& \quad \left. 2c_1 c_2 - c_2^2, 3c_2^2) \right] \{\theta_0(1-\theta_0)\}^3 \sum^{(15)} (c_1^{-1}, 0, 0, c_2^{-1})^{\langle 3 \rangle}.
\end{aligned}$$

Then,

$$\begin{aligned}
\kappa_4(t_\eta) &= n^{-1} \left[\left(\frac{1}{c_1 c_2} - 3 \right) \left(\frac{1}{\theta_0(1-\theta_0)} - 6 \right) \right. \\
& \quad - (c_1 c_2)^2 \frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)} \{(-1,1)^{\langle 3 \rangle} \otimes (-2c_1, c_1 - c_2, c_1 - c_2, 2c_2)\} \\
& \quad \times \sum^{(10)} (c_1^{-1}, 0, 0, c_2^{-1})' \otimes (c_1^{-2}, \mathbf{0}'_{(6)}, c_2^{-2}) \\
& \quad \left[\frac{3}{8} \frac{(c_1 c_2)^2}{\theta_0(1-\theta_0)} (1-2\theta_0)^2 \{(-1,1)^{\langle 2 \rangle} \otimes (-2c_1, c_1 - c_2, c_1 - c_2, 2c_2)^{\langle 2 \rangle}\} \right. \\
& \quad + \frac{2}{3} (c_1 c_2)^2 \left\{ 1 + \frac{3}{4} \frac{(1-2\theta_0)^2}{\theta_0(1-\theta_0)} \right\} \\
& \quad \times (-1,1)^{\langle 3 \rangle} \otimes (-3c_1^2, c_1^2 - 2c_1 c_2, c_1^2 - 2c_1 c_2, 2c_1 c_2 - c_2^2, c_1^2 - 2c_1 c_2, 2c_1 c_2 - c_2^2, \\
& \quad \quad \left. 2c_1 c_2 - c_2^2, 3c_2^2) \right] \sum^{(15)} (c_1^{-1}, 0, 0, c_2^{-1})^{\langle 3 \rangle} - 6\alpha_{\eta\Delta 2}^{(t)} \left. \right] + O(n^{-2}) \\
& \equiv n^{-1} \alpha_{\eta 4}^{(t)} + O(n^{-2}).
\end{aligned}$$

$$n \text{acov}(n^{-1} \hat{\eta}, \hat{\alpha}_{\eta 1}^{(t)}) = 0 \quad (\text{recall that } \alpha_{\eta 1}^{(t)} = 0),$$

$$\begin{aligned}
n \operatorname{acov}(n^{-1} \hat{\eta}, \hat{\alpha}_{\eta_3}^{(t)}) &= (c_1 c_2)^{1/2} (c_1 - c_2) n \operatorname{acov} \left[\frac{\hat{p}_2 - \hat{p}_1}{\hat{\theta}(1 - \hat{\theta})}, \frac{1 - 2\hat{\theta}}{\{\hat{\theta}(1 - \hat{\theta})\}^{1/2}} \right] \\
&= \frac{(c_1 c_2)^{1/2}}{\theta_0(1 - \theta_0)} (c_1 - c_2) \left[-\frac{2}{\{\theta_0(1 - \theta_0)\}^{1/2}} - \frac{(1 - 2\theta_0)^2}{2\{\theta_0(1 - \theta_0)\}^{3/2}} \right] \\
&\quad \times \theta_0(1 - \theta_0)(-1, 1) \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
&= 0.
\end{aligned}$$

4.4 Asymptotic cumulants of $\hat{\theta}_W$ and $\hat{\eta}_W$ by the weighted score method

$$\begin{aligned}
\hat{\theta}_W &= \frac{m_1 + 2k}{n + 4k} = \frac{\hat{\theta} + n^{-1}2k}{1 + n^{-1}4k} = \hat{\theta} \left(1 - \frac{n^{-1}4k}{1 + n^{-1}4k} \right) + \frac{n^{-1}2k}{1 + n^{-1}4k} \\
&= \hat{\theta} - n^{-1}4k\hat{\theta} + n^{-1}2k + O_p(n^{-2}) = \hat{\theta} + n^{-1}2k(1 - 2\hat{\theta}) + O_p(n^{-2}).
\end{aligned}$$

In the following, only the asymptotic cumulants different from those by ML are shown.

$$\begin{aligned}
\kappa_1(\hat{\theta}_W - \theta_0) &= n^{-1}\alpha_{\theta_1} + n^{-1}2k(1 - 2\theta_0) + O(n^{-2}) \\
&= n^{-1}2k(1 - 2\theta_0) + O(n^{-2}) \\
&\equiv n^{-1}\alpha_{W\theta_1} + O(n^{-2}) \quad (\text{recall that } \alpha_{\theta_1} = 0), \\
\kappa_2(\hat{\theta}_W) &= n^{-1}\alpha_{\theta_2} + n^{-2}(\alpha_{\theta\Delta_2} - 8k\alpha_{\theta_2}) + O(n^{-2}) \\
&\equiv n^{-1}\alpha_{\theta_2} + n^{-2}\alpha_{W\theta\Delta_2} + O(n^{-3}) \quad (\alpha_{\theta_2} = \alpha_{W\theta_2}; \alpha_{W\theta\Delta_2} \leq \alpha_{\theta\Delta_2}).
\end{aligned}$$

$$\hat{\eta}_W = \frac{n_W c_{W1} c_{W2} (\hat{p}_{W2} - \hat{p}_{W1})}{\hat{\theta}_W (1 - \hat{\theta}_W)}, \text{ where}$$

$$n_W \equiv n + 4k, c_{W1} = \frac{n_1 + 2k}{n + 4k}, c_{W2} = \frac{n_2 + 2k}{n + 4k},$$

$$\hat{p}_{W1} = \frac{m_{11} + k}{n_1 + 2k}, \hat{p}_{W2} \equiv \frac{m_{21} + k}{n_2 + 2k},$$

$$c_{w1} = \frac{c_1 + n^{-1}2k}{1 + n^{-1}4k} = c_1 + n^{-1}2k(1 - 2c_1) + O(n^{-2}),$$

$$c_{w2} = c_2 + n^{-1}2k(1 - 2c_2) + O(n^{-2}),$$

$$\hat{p}_{w1} = \frac{\hat{p}_1 + n^{-1}k}{1 + n^{-1}2k} = \frac{\hat{p}_1 + n^{-1}c_1^{-1}k}{1 + n^{-1}c_1^{-1}2k} = \hat{p}_1 + n^{-1}c_1^{-1}k(1 - 2\hat{p}_1) + O_p(n^{-2}),$$

$$\hat{p}_{w2} = \hat{p}_2 + n^{-1}c_2^{-1}k(1 - 2\hat{p}_2) + O_p(n^{-2}),$$

$$\hat{p}_{w2} - \hat{p}_{w1} = \hat{p}_2 - \hat{p}_1 + n^{-1}k\{c_2^{-1} - c_1^{-1} - 2(c_2^{-1}\hat{p}_2 - c_1^{-1}\hat{p}_1)\} + O_p(n^{-2}),$$

$$\begin{aligned} \{\hat{\theta}_w(1 - \hat{\theta}_w)\}^{-1} &= \{\hat{\theta}(1 - \hat{\theta})\}^{-1} - \{\hat{\theta}(1 - \hat{\theta})\}^{-2}(1 - 2\hat{\theta})(\hat{\theta}_w - \hat{\theta}) + O_p(n^{-2}) \\ &= \{\hat{\theta}(1 - \hat{\theta})\}^{-1} - n^{-1}\{\hat{\theta}(1 - \hat{\theta})\}^{-2}2k(1 - 2\hat{\theta})^2 + O_p(n^{-2}). \end{aligned}$$

From the above results,

$$\begin{aligned} n^{-1}\hat{\eta}_w &= n^{-1}(n + 4k)\{c_1 + n^{-1}2k(1 - 2c_1)\}\{c_2 + n^{-1}2k(1 - 2c_2)\} \\ &\quad \times [\hat{p}_2 - \hat{p}_1 + n^{-1}k\{c_2^{-1} - c_1^{-1} - 2(c_2^{-1}\hat{p}_2 - c_1^{-1}\hat{p}_1)\}] \\ &\quad \times \left[\frac{1}{\hat{\theta}(1 - \hat{\theta})} - n^{-1} \frac{2k(1 - 2\hat{\theta})^2}{\{\hat{\theta}(1 - \hat{\theta})\}^2} \right] + O_p(n^{-2}) \\ &= \frac{c_1c_2(\hat{p}_2 - \hat{p}_1)}{\hat{\theta}(1 - \hat{\theta})} \\ &\quad + n^{-1} \left[\left\{ 4k + 2kc_1^{-1}(1 - 2c_1) + 2kc_2^{-1}(1 - 2c_2) - \frac{2k(1 - 2\hat{\theta})^2}{\hat{\theta}(1 - \hat{\theta})} \right\} \right. \\ &\quad \left. \times \frac{c_1c_2(\hat{p}_2 - \hat{p}_1)}{\hat{\theta}(1 - \hat{\theta})} + k\{c_2^{-1} - c_1^{-1} - 2(c_2^{-1}\hat{p}_2 - c_1^{-1}\hat{p}_1)\} \frac{c_1c_2}{\hat{\theta}(1 - \hat{\theta})} \right] + O_p(n^{-2}) \\ &= n^{-1}\hat{\eta} + n^{-1} \left[\left\{ -4k + \frac{2k}{c_1c_2} - \frac{2k(1 - 2\hat{\theta})^2}{\hat{\theta}(1 - \hat{\theta})} \right\} n^{-1}\hat{\eta} \right. \\ &\quad \left. + k\{c_1 - c_2 - 2(c_1\hat{p}_2 - c_2\hat{p}_1)\} \frac{1}{\hat{\theta}(1 - \hat{\theta})} \right] + O_p(n^{-2}). \end{aligned}$$

Then,

$$\begin{aligned}
\kappa_1(n^{-1}\hat{\eta}_W - n^{-1}\eta_0) &= n^{-1} \left\{ \alpha_{\eta_1} + k \frac{(c_1 - c_2)(1 - 2\theta_0)}{\theta_0(1 - \theta_0)} \right\} + O(n^{-2}) \\
&= n^{-1} k \frac{(c_1 - c_2)(1 - 2\theta_0)}{\theta_0(1 - \theta_0)} + O(n^{-2}) \\
&\equiv n^{-1} \alpha_{W\eta_1} + O(n^{-2})
\end{aligned}$$

(recall that $\eta_0 = 0$ and $\alpha_{\eta_1} = 0$),

$$\begin{aligned}
\kappa_2(n^{-1}\hat{\eta}_W) &= n^{-1} \alpha_{\eta_2} + n^{-2} \left[\alpha_{\eta_{\Delta 2}} + 4k \left\{ -2 + \frac{1}{c_1 c_2} - \frac{(1 - 2\theta_0)^2}{\theta_0(1 - \theta_0)} \right\} \alpha_{\eta_2} \right. \\
&\quad - 2k \frac{(c_1 - c_2)(1 - 2\theta_0)^2}{\{\theta_0(1 - \theta_0)\}^2} \theta_0(1 - \theta_0)(c_1, c_2) \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix} \frac{\partial n^{-1}\eta_0}{\partial \mathbf{p}} \\
&\quad \left. - 4k \frac{\partial(c_1 p_2 - c_2 p_1)}{\partial \mathbf{p}} \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix} \frac{\partial n^{-1}\eta_0}{\partial \mathbf{p}} \right] + O(n^{-3}) \\
&= n^{-1} \alpha_{\eta_2} + n^{-2} \left[\underset{(A)}{\alpha_{\eta_{\Delta 2}}} + 4k \left[\underset{(B)}{\left\{ -2 + \frac{1}{c_1 c_2} - \frac{(1 - 2\theta_0)^2}{\theta_0(1 - \theta_0)} \right\}} \alpha_{\eta_2} \right. \right. \\
&\quad \left. \left. - \frac{1 - 2c_1 c_2}{\theta_0(1 - \theta_0)} \right] \right] + O(n^{-3}) \\
&\equiv n^{-1} \alpha_{\eta_2} + n^{-2} \alpha_{W\eta_{\Delta 2}} + O(n^{-3}) \quad (\alpha_{W\eta_2} = \alpha_{\eta_2}),
\end{aligned}$$

where $(c_1, c_2) \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix} \frac{\partial n^{-1}\eta_0}{\partial \mathbf{p}} = 0$ and

$$\begin{aligned}
\frac{\partial(c_1 p_2 - c_2 p_1)}{\partial \mathbf{p}} \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix} \frac{\partial n^{-1}\eta_0}{\partial \mathbf{p}} &= (-c_2, c_1) \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \frac{c_1 c_2}{\theta_0(1 - \theta_0)} \\
&= \left(\frac{c_2}{c_1} + \frac{c_1}{c_2} \right) \frac{c_1 c_2}{\theta_0(1 - \theta_0)} = \frac{1 - 2c_1 c_2}{\theta_0(1 - \theta_0)}
\end{aligned}$$

are used with $\frac{\partial n^{-1}\eta_0}{\partial \mathbf{p}} = \frac{c_1 c_2}{\theta_0(1 - \theta_0)} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

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