# An expository supplement to the paper "Bias adjustment minimizing the asymptotic mean square error" with errata

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This expository article is to supplement Ogasawara (2013) with errata. Section 1 gives the asymptotic bias and added higher-order asymptotic variance of the normally distributed variable with a known coefficient of variation. This distribution is one of the simplest cases in the curved exponential family. Section 2 gives errata for the online version of Ogasawara (2013) and the corresponding print version unless corrected at the time of publication.

## 1. Supplement

Let $X_i \overset{\text{i.i.d.}}{\sim} N(\mu, c\mu^2)$ $(i = 1, ..., n)$, $\mu > 0$. For simplicity, $c$ is set to be 1 as in Firth (1993). Define $L = \prod_{i=1}^{n} \dfrac{1}{\sqrt{2\pi}\mu} \exp\left\{ -\dfrac{(x_i - \mu)^2}{2\mu^2} \right\}$ and $l = \log L$, then

$$\frac{\partial l}{\partial \mu} = \frac{\sum_{i=1}^{n} x_i^2}{\mu^3} - \frac{\sum_{i=1}^{n} x_i}{\mu^2} - \frac{n}{\mu},$$

$$-\frac{\partial^2 l}{\partial \mu^2} = \frac{3 \sum_{i=1}^{n} x_i^2}{\mu^4} - \frac{2 \sum_{i=1}^{n} x_i}{\mu^3} - \frac{n}{\mu^2}$$

and the Fisher information per observation becomes

$$i_{(\mu)} = \frac{3(\mu^2 + \mu^2)}{\mu^4} - \frac{2\mu}{\mu^3} - \frac{1}{\mu^2} = \frac{3}{\mu^2}.$$

Define $u \equiv n^{-1} \sum_{i=1}^{n} x_i^2$, $m \equiv n^{-1} \sum_{i=1}^{n} x_i$, $\mathbf{y} \equiv (u, m)'$, $\mathrm{E}(\mathbf{y}) = (2\mu^2, \mu)' \equiv \zeta$. Let the first-order condition of the estimator $\hat{\mu}$ be

$$f \equiv n^{-1}\mu^3 \frac{\partial l}{\partial \mu} = u - m\mu - \mu^2 = 0.$$

Then,

$$\frac{\partial f}{\partial \mu} = -m - 2\mu, \quad \frac{\partial^2 f}{\partial \mu^2} = -2, \quad \frac{\partial^3 f}{\partial \mu^3} = 0, \quad \frac{\partial f}{\partial \mathbf{y}} = (1, -\mu)',$$

$$\frac{\partial^2 f}{\partial \mu \partial \mathbf{y}} = (0, -1)', \frac{\partial^2 f}{\partial \mu^2 \partial \mathbf{y}} = (0, 0)',$$

$$\frac{\partial \mu}{\partial \boldsymbol{\zeta}} \equiv \frac{\partial \hat{\mu}}{\partial \mathbf{y}}\Big|_{\mathbf{y}=\boldsymbol{\zeta}} = -\left(\frac{\partial f}{\partial \mu}\right)^{-1}\frac{\partial f}{\partial \boldsymbol{\zeta}} = (\mu + 2\mu)^{-1}(1, -\mu)' = \frac{1}{3}(\mu^{-1}, -1)',$$

$$\frac{\partial^2 \mu}{\partial \boldsymbol{\zeta} \partial \boldsymbol{\zeta}'} = -\left(\frac{\partial f}{\partial \mu}\right)^{-1}\left(\frac{\partial^2 f}{\partial \mu^2}\frac{\partial \mu}{\partial \boldsymbol{\zeta}}\frac{\partial \mu}{\partial \boldsymbol{\zeta}'} + \frac{\partial^2 f}{\partial \mu \partial \boldsymbol{\zeta}}\frac{\partial \mu}{\partial \boldsymbol{\zeta}'} + \frac{\partial \mu}{\partial \boldsymbol{\zeta}}\frac{\partial^2 f}{\partial \mu \partial \boldsymbol{\zeta}'}\right)$$

$$= (3\mu)^{-1}\left\{(-2)\frac{1}{9}\binom{\mu^{-1}}{-1}(\mu^{-1}, -1) + \binom{0}{-1}\frac{1}{3}(\mu^{-1}, -1) + \frac{1}{3}\binom{\mu^{-1}}{-1}(0, -1)\right\}$$

$$= (3\mu)^{-1}\left\{-\frac{2}{9}\begin{pmatrix} \mu^{-2} & -\mu^{-1} \\ -\mu^{-1} & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\mu^{-1}/3 & 1/3 \end{pmatrix} + \begin{pmatrix} 0 & -\mu^{-1}/3 \\ 0 & 1/3 \end{pmatrix}\right\}$$

$$= (3\mu)^{-1}\begin{pmatrix} -\dfrac{2}{9}\mu^{-2} & -\dfrac{1}{9}\mu^{-1} \\ -\dfrac{1}{9}\mu^{-1} & \dfrac{4}{9} \end{pmatrix} = \frac{1}{27}\begin{pmatrix} -2\mu^{-3} & -\mu^{-2} \\ -\mu^{-2} & 4\mu^{-1} \end{pmatrix},$$

$$\frac{\partial^3 \mu}{\partial \boldsymbol{\zeta} \partial \boldsymbol{\zeta}' \partial \boldsymbol{\zeta}_i} = -\left(\frac{\partial f}{\partial \mu}\right)^{-1}\left\{\frac{\partial^3 f}{\partial \mu^3}\frac{\partial \mu}{\partial \boldsymbol{\zeta}}\frac{\partial \mu}{\partial \boldsymbol{\zeta}'}\frac{\partial \mu}{\partial \boldsymbol{\zeta}_i}\right.$$

$$+ \frac{\partial^2 f}{\partial \mu^2}\left(\frac{\partial \mu}{\partial \boldsymbol{\zeta}}\frac{\partial^2 \mu}{\partial \boldsymbol{\zeta}' \partial \boldsymbol{\zeta}_i} + \frac{\partial^2 \mu}{\partial \boldsymbol{\zeta} \partial \boldsymbol{\zeta}_i}\frac{\partial \mu}{\partial \boldsymbol{\zeta}'} + \frac{\partial^2 \mu}{\partial \boldsymbol{\zeta} \partial \boldsymbol{\zeta}'}\frac{\partial \mu}{\partial \boldsymbol{\zeta}_i}\right)$$

$$+ \frac{\partial^3 f}{\partial \mu^2 \partial \zeta} \frac{\partial \mu}{\partial \zeta'} \frac{\partial \mu}{\partial \zeta_i} + \frac{\partial \mu}{\partial \zeta} \frac{\partial^3 f}{\partial \mu^2 \partial \zeta'} \frac{\partial \mu}{\partial \zeta_i} + \frac{\partial^3 f}{\partial \mu^2 \partial \zeta_i} \frac{\partial \mu}{\partial \zeta} \frac{\partial \mu}{\partial \zeta'}$$

$$+ \frac{\partial^2 f}{\partial \mu \partial \zeta} \frac{\partial^2 \mu}{\partial \zeta' \partial \zeta_i} + \frac{\partial^2 \mu}{\partial \zeta \partial \zeta_i} \frac{\partial^2 f}{\partial \mu \partial \zeta'} + \frac{\partial^2 f}{\partial \mu \partial \zeta_i} \frac{\partial^2 \mu}{\partial \zeta \partial \zeta'} \Bigg\}$$

$$= -\left( \frac{\partial f}{\partial \mu} \right)^{-1} \Bigg\{ \frac{\partial^2 f}{\partial \mu^2} \left( \frac{\partial \mu}{\partial \zeta} \frac{\partial^2 \mu}{\partial \zeta' \partial \zeta_i} + \frac{\partial^2 \mu}{\partial \zeta \partial \zeta_i} \frac{\partial \mu}{\partial \zeta'} + \frac{\partial^2 \mu}{\partial \zeta \partial \zeta'} \frac{\partial \mu}{\partial \zeta_i} \right)$$

$$+ \frac{\partial^2 f}{\partial \mu \partial \zeta} \frac{\partial^2 \mu}{\partial \zeta' \partial \zeta_i} + \frac{\partial^2 \mu}{\partial \zeta \partial \zeta_i} \frac{\partial^2 f}{\partial \mu \partial \zeta'} + \frac{\partial^2 f}{\partial \mu \partial \zeta_i} \frac{\partial^2 \mu}{\partial \zeta \partial \zeta'} \Bigg\} \quad (i = 1, 2),$$

$$\frac{\partial^3 \mu}{\partial \zeta \partial \zeta' \partial \zeta_1} = (3\mu)^{-1} \left[ -2 \frac{1}{81} \left\{ \begin{pmatrix} \mu^{-1} \\ -1 \end{pmatrix} (-2\mu^{-3}, \ -\mu^{-2}) + \begin{pmatrix} -2\mu^{-3} \\ -\mu^{-2} \end{pmatrix} (\mu^{-1}, -1) \right. \right.$$

$$+ \begin{pmatrix} -2\mu^{-3} & -\mu^{-2} \\ -\mu^{-2} & 4\mu^{-1} \end{pmatrix} \mu^{-1} \Bigg\} + \frac{1}{27} \left\{ \begin{pmatrix} 0 \\ -1 \end{pmatrix} (-2\mu^{-3}, \ -\mu^{-2}) + \begin{pmatrix} -2\mu^{-3} \\ -\mu^{-2} \end{pmatrix} (0, -1) \right\} \Bigg]$$

$$= (3\mu)^{-1} \left\{ -\frac{2}{81} \begin{pmatrix} -6\mu^{-4} & 0 \\ 0 & 6\mu^{-2} \end{pmatrix} + \frac{1}{27} \begin{pmatrix} 0 & 2\mu^{-3} \\ 2\mu^{-3} & 2\mu^{-2} \end{pmatrix} \right\}$$

$$= (3\mu)^{-1} \frac{1}{27} \begin{pmatrix} 4\mu^{-4} & 2\mu^{-3} \\ 2\mu^{-3} & -2\mu^{-2} \end{pmatrix} = \frac{1}{81} \begin{pmatrix} 4\mu^{-5} & 2\mu^{-4} \\ 2\mu^{-4} & -2\mu^{-3} \end{pmatrix},$$

$$\frac{\partial^3 \mu}{\partial \zeta \partial \zeta' \partial \zeta_2} = (3\mu)^{-1} \left[ -2\frac{1}{81} \left\{ \begin{pmatrix} \mu^{-1} \\ -1 \end{pmatrix} (-\mu^{-2}, 4\mu^{-1}) + \begin{pmatrix} -\mu^{-2} \\ 4\mu^{-1} \end{pmatrix} (\mu^{-1}, -1) \right. \right.$$

$$\left. + \begin{pmatrix} -2\mu^{-3} & -\mu^{-2} \\ -\mu^{-2} & 4\mu^{-1} \end{pmatrix} (-1) \right\}$$

$$\left. + \frac{1}{27} \left\{ \begin{pmatrix} 0 \\ -1 \end{pmatrix} (-\mu^{-2}, 4\mu^{-1}) + \begin{pmatrix} -\mu^{-2} \\ 4\mu^{-1} \end{pmatrix} (0, -1) + (-1) \begin{pmatrix} -2\mu^{-3} & -\mu^{-2} \\ -\mu^{-2} & 4\mu^{-1} \end{pmatrix} \right\} \right]$$

$$= (3\mu)^{-1} \left\{ -\frac{2}{81} \begin{pmatrix} 0 & 6\mu^{-2} \\ 6\mu^{-2} & -12\mu^{-1} \end{pmatrix} + \frac{1}{27} \begin{pmatrix} 2\mu^{-3} & 2\mu^{-2} \\ 2\mu^{-2} & -12\mu^{-1} \end{pmatrix} \right\}$$

$$= (3\mu)^{-1} \frac{1}{27} \begin{pmatrix} 2\mu^{-3} - 2\mu^{-2} \\ -2\mu^{-2} - 4\mu^{-1} \end{pmatrix} = \frac{1}{81} \begin{pmatrix} 2\mu^{-4} - 2\mu^{-3} \\ -2\mu^{-3} - 4\mu^{-2} \end{pmatrix}.$$

Expand $\hat{\mu}$ about $\mu$ as

$$\hat{\mu} = \mu + \frac{\partial \mu}{\partial \zeta'} (\mathbf{y} - \boldsymbol{\zeta}) + \frac{1}{2} \frac{\partial^2 \mu}{(\partial \zeta')^{<2>}} (\mathbf{y} - \boldsymbol{\zeta})^{<2>} + \frac{1}{6} \frac{\partial^3 \mu}{(\partial \zeta')^{<3>}} (\mathbf{y} - \boldsymbol{\zeta})^{<3>}$$

$$+ O_p(n^{-2}).$$

From

$$n \, \mathrm{E}\{(u - \mathrm{E}(u))^2\} = \mathrm{E}\{(X_i^2 - \mathrm{E}(X_i^2))^2\}$$

$$= \mathrm{E}[\{(X_i - \mu)^2 + 2\mu(X_i - \mu) - \mu^2\}^2]$$

$$= 3\mu^4 + 4\mu^4 + \mu^4 - 2\mu^4 = 6\mu^4,$$

$$n \, \mathrm{E}\{(u - \mathrm{E}(u))(m - \mathrm{E}(m))\} = \mathrm{E}\{2\mu(X_i - \mu))(X_i - \mu)\} = 2\mu^3,$$

$$n \, \mathrm{E}\{(m - \mathrm{E}(m))^2\} = \mu^2,$$

we have

$$n \, \mathrm{E}\{(\mathbf{y} - \boldsymbol{\zeta})(\mathbf{y} - \boldsymbol{\zeta})'\} = \begin{pmatrix} 6\mu^4 & 2\mu^3 \\ 2\mu^3 & \mu^2 \end{pmatrix} \equiv \boldsymbol{\Omega} \quad \text{and}$$

$$E(\hat{\mu}) = n^{-1}\frac{1}{2}\frac{\partial^2\mu}{(\partial\varsigma)^{<2>}}E\{(\mathbf{y}-\varsigma)^{<2>}\}+O(n^{-2})$$

$$= n^{-1}\frac{1}{54}\operatorname{tr}\left\{\begin{pmatrix}-2\mu^{-3}-\mu^{-2} & \\ -\mu^{-2} & 4\mu^{-1}\end{pmatrix}\begin{pmatrix}6\mu^4 & 2\mu^3 \\ 2\mu^3 & \mu^2\end{pmatrix}\right\}+O(n^{-2})$$

$$= n^{-1}\frac{\mu}{54}(-12-2-2+4)+O(n^{-2}) = -n^{-1}\frac{2}{9}\mu+O(n^{-2}),$$

which is found to be equal to the result of Firth (1993, p.35)

   Further.

$$n^2\,E\{(u-E(u))^3\} = E[\{(X_i-\mu)^2-\mu^2+2\mu(X_i-\mu)\}^3]$$

$$= E[\{(X_i-\mu)^2-\mu^2\}^3]+6\mu E[\{(X_i-\mu)^2-\mu^2\}^2(X_i-\mu)]$$

$$+12\mu^2 E[\{(X_i-\mu)^2-\mu^2\}(X_i-\mu)^2]+8\mu^3 E\{(X_i-\mu)^3\}$$

$$= (15-3\times3+3-1)\mu^6+12\mu^2(3-1)\mu^4 = 32\mu^6,$$

where $n^2 E\{\cdot\}$ comes from the following properties: (1) the cumulant of the third power of a sample mean is $n^{-3}$ times that of the sum of independent $X_i\,(i=1,...,n)$ and (2) the cumulant of the sum is $n$ times that of $X_i$.

$$n^2\,E\{(u-E(u))^2(m-E(m))\}$$

$$= E[\{(X_i-\mu)^2-\mu^2+2\mu(X_i-\mu)\}^2(X_i-\mu)]$$

$$= 4\mu E[\{(X_i-\mu)^2-\mu^2\}(X_i-\mu)^2] = 4\mu(3-1)\mu^4$$

$$= 8\mu^5,$$

$$n^2\,E\{(u-E(u))(m-E(m))^2\}$$

$$= E[\{(X_i-\mu)^2-\mu^2+2\mu(X_i-\mu)\}(X_i-\mu)^2] = (3-1)\mu^4$$

$$= 2\mu^4,$$

$$n^2\,E\{(m-E(m))^3\} = 0.$$

   The variance of $\hat{\mu}$ is

$$\mathrm{var}(\hat{\mu}) = n^{-1}\frac{\partial \mu}{\partial \zeta'}\boldsymbol{\Omega}\frac{\partial \mu}{\partial \zeta} + n^{-2}\left[\frac{1}{2}\mathrm{tr}\left(\frac{\partial^2 \mu}{\partial \zeta \partial \zeta'}\boldsymbol{\Omega}\frac{\partial^2 \mu}{\partial \zeta \partial \zeta'}\boldsymbol{\Omega}\right)\right.$$

$$+\left\{\frac{\partial \mu}{\partial \zeta'}\otimes\frac{\partial^2 \mu}{(\partial \zeta')^{<2>}}\right\}n^2 \mathrm{E}\{(\mathbf{y}-\boldsymbol{\zeta})^{<3>}\}$$

$$+\frac{1}{3}\left\{\frac{\partial \mu}{\partial \zeta'}\otimes\frac{\partial^3 \mu}{(\partial \zeta')^{<3>}}\right\}n^2 \mathrm{E}\{(\mathbf{y}-\boldsymbol{\zeta})^{<4>}\}\right]+O_p(n^{-3}),$$

where the first term is

$$= n^{-1}\frac{1}{9}(\mu^{-1},-1)\begin{pmatrix}6\mu^4 & 2\mu^3 \\ 2\mu^3 & \mu^2\end{pmatrix}\begin{pmatrix}\mu^{-1} \\ -1\end{pmatrix}= n^{-1}\frac{1}{9}(\mu^{-1},-1)(4\mu^3,\mu^2)'$$

$$= n^{-1}\frac{\mu^2}{3},$$

which is equal to the reciprocal of the Fisher information. The first term in $n^{-2}[\cdot]$ is

$$=\frac{1}{2}\times\frac{1}{27^2}\mathrm{tr}\left\{\begin{pmatrix}2\mu^{-1} & 1 \\ 1 & -4\mu\end{pmatrix}\begin{pmatrix}6\mu^2 & 2\mu \\ 2\mu & 1\end{pmatrix}\begin{pmatrix}2\mu^{-1} & 1 \\ 1 & -4\mu\end{pmatrix}\begin{pmatrix}6\mu^2 & 2\mu \\ 2\mu & 1\end{pmatrix}\right\}$$

$$=\frac{1}{2}\times\frac{1}{27^2}\mathrm{tr}\left\{\begin{pmatrix}14\mu & 5 \\ -2\mu^2 & -2\mu\end{pmatrix}\begin{pmatrix}14\mu & 5 \\ -2\mu^2 & -2\mu\end{pmatrix}\right\}$$

$$=\frac{1}{2}\times\frac{1}{27^2}\{14^2-5\times2-2\times5+(-2)^2\}\mu^2=\frac{90}{27^2}\mu^2=\frac{10}{81}\mu^2.$$

The second term in $n^{-2}[\cdot]$ is

$$= \frac{\partial \mu}{\partial \zeta_1} \frac{\partial^2 \mu}{\partial \zeta_1^2} n^2 E\{(u - E(u))^3\}$$

$$+ \left( 2 \frac{\partial \mu}{\partial \zeta_1} \frac{\partial^2 \mu}{\partial \zeta_1 \partial \zeta_2} + \frac{\partial \mu}{\partial \zeta_2} \frac{\partial^2 \mu}{\partial \zeta_1^2} \right) n^2 E\{(u - E(u))^2 (m - E(m))\}$$

$$+ \left( 2 \frac{\partial \mu}{\partial \zeta_2} \frac{\partial^2 \mu}{\partial \zeta_1 \partial \zeta_2} + \frac{\partial \mu}{\partial \zeta_1} \frac{\partial^2 \mu}{\partial \zeta_2^2} \right) n^2 E\{(u - E(u))(m - E(m))^2\}$$

$$+ \frac{\partial \mu}{\partial \zeta_2} \frac{\partial^2 \mu}{\partial \zeta_2^2} n^2 E\{(m - E(m))^3\}$$

$$= \frac{\mu^{-1}}{3} \frac{-2\mu^{-3}}{27} 32\mu^6 + \left( 2 \times \frac{\mu^{-1}}{3} \frac{-\mu^{-2}}{27} + \frac{-1}{3} \times \frac{-2\mu^{-3}}{27} \right) 8\mu^5$$

$$+ \left( 2 \times \frac{-1}{3} \times \frac{-\mu^{-2}}{27} + \frac{\mu^{-1}}{3} \times \frac{4\mu^{-1}}{27} \right) 2\mu^4$$

$$= \left( -\frac{64}{81} + \frac{6 \times 2}{81} \right) \mu^2 = -\frac{52}{81} \mu^2.$$

Recall $\boldsymbol{\Omega} = \begin{pmatrix} 6\mu^4 & 2\mu^3 \\ 2\mu^3 & \mu^2 \end{pmatrix}$. Then, the third term in $n^{-2}[\cdot]$ is

$$= \left\{ \frac{\partial \mu}{\partial \zeta'} \otimes \frac{\partial^3 \mu}{(\partial \zeta')^{<3>}} \right\} \{vec(\boldsymbol{\Omega})\}^{<2>}$$

$$= \frac{\partial \mu}{\partial \zeta_1} \frac{\partial^3 \mu}{\partial \zeta_1^3} \omega_{11}^2 + \left( 3 \frac{\partial \mu}{\partial \zeta_1} \frac{\partial^3 \mu}{\partial \zeta_1^2 \partial \zeta_2} + \frac{\partial \mu}{\partial \zeta_2} \frac{\partial^3 \mu}{\partial \zeta_1^3} \right) \omega_{11} \omega_{12}$$

$$+\left(3\frac{\partial\mu}{\partial\zeta_2}\frac{\partial^3\mu}{\partial\zeta_2^2\partial\zeta_1}+\frac{\partial\mu}{\partial\zeta_1}\frac{\partial^3\mu}{\partial\zeta_2^3}\right)\omega_{22}\omega_{12}$$

$$+\left(\frac{\partial\mu}{\partial\zeta_1}\frac{\partial^3\mu}{\partial\zeta_1\partial\zeta_2^2}+\frac{\partial\mu}{\partial\zeta_2}\frac{\partial^3\mu}{\partial\zeta_2\partial\zeta_1^2}\right)(\omega_{11}\omega_{22}+2\omega_{12}^2)$$

$$+\frac{\partial\mu}{\partial\zeta_2}\frac{\partial^3\mu}{\partial\zeta_2^3}\omega_{22}^2$$

$$=\frac{\mu^{-1}}{3}\frac{4\mu^{-5}}{81}(6\mu^4)^2+\left(3\times\frac{\mu^{-1}}{3}\frac{2\mu^{-4}}{81}+\frac{-1}{3}\frac{4\mu^{-5}}{81}\right)6\mu^4 2\mu^3$$

$$+\left(3\times\frac{-1}{3}\frac{-2\mu^{-3}}{81}+\frac{\mu^{-1}}{3}\frac{-4\mu^{-2}}{81}\right)\mu^2 2\mu^3$$

$$+\left(\frac{\mu^{-1}}{3}\frac{-2\mu^{-3}}{81}+\frac{-1}{3}\frac{2\mu^{-4}}{81}\right)\{6\mu^4\mu^2+2(2\mu^3)^2\}+\frac{-1}{3}\frac{-4\mu^{-4}}{81}(\mu^2)^2$$

$$=\left\{\frac{48}{81}+\left(\frac{2}{81}-\frac{4}{3\times81}\right)12+\left(\frac{2}{81}-\frac{4}{3\times81}\right)2+\left(\frac{-4}{3\times81}\right)14+\frac{4}{3\times81}\right\}\mu^2$$

$$=\frac{1}{3\times81}(48\times3+2\times12+2\times2-56+4)\mu^2=\frac{1}{243}(144+32-56)\mu^2$$

$$=\frac{120}{243}\mu^2=\frac{40}{81}\mu^2.$$

The sum of the terms inside $n^{-2}[\cdot]$ becomes

$$=\left(\frac{10}{81}-\frac{52}{81}+\frac{40}{81}\right)\mu^2=-\frac{2}{81}\mu^2,$$

which gives

$$\operatorname{var}(\hat{\mu})=n^{-1}\mu^2-n^{-2}\frac{2}{81}\mu^2+O(n^{-3})$$

equal to the result of Firth (1993, p.35).

## 2. Errata

In Theorem1, the phrase "and $\beta_{\Delta 1}^{(U)} = 0$" at the end of the sentence should be deleted. That is, the theorem becomes

**Theorem 1**. *The* $\mathrm{MSE}(c_{\min}^{*(U)}\hat{\theta}^{(U)})$ *up to order* $O(n^{-2})$ *is greater than or equal to (smaller than) the corresponding* $\mathrm{MSE}(\hat{\theta}_{Ck\min})$ *when*

$$\theta_0^{-2}\beta_2^2 \leq (>) \; \beta_1^{-2}\{\mathrm{acov}(\hat{\theta}, \hat{\beta}_1)\}^2 \text{, where the equality occurs when}$$

$$\beta_1 / \theta_0 = (\hat{\beta}_1 / \hat{\theta}) + O(n^{-1}) = O(1).$$

In Subsection 3.1 Mean, the expression " $= \dfrac{n^{-1}\sigma^2}{1 + n^{-1}(c_V^{(L)})^2}$ " in the last equation should be deleted. That is, the equation should be

$$\mathrm{MSE}(c_{(L)CV}^*\hat{\theta}^*) < \mathrm{MSE}(\hat{\theta}^*) = \mathrm{MSE}(\bar{X}).$$

Equation (5.8) and the sentence including the equation should be as follows:

$$\hat{\boldsymbol{\theta}}_W - \boldsymbol{\theta}_0$$

$$= -n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^* + \sum_{i=1}^{3}\boldsymbol{\Lambda}^{(i)}\mathbf{l}_0^{(i)} + n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^* - n^{-1}\boldsymbol{\Lambda}^{-1}\frac{\partial \mathbf{q}_0^*}{\partial \boldsymbol{\theta}_0{'}}\boldsymbol{\Lambda}^{(1)}\mathbf{l}_0^{(1)}$$

$$- n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{J}_0^{(3)}\{(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*) \otimes \boldsymbol{\Lambda}^{-1}\}\mathbf{l}_0^{(1)} + O_p(n^{-2}) \tag{5.8}$$

$$= -n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^* + \sum_{i=1}^{3}\boldsymbol{\Lambda}^{(i)}\mathbf{l}_0^{(i)} - n^{-1}(\hat{\mathbf{L}}^{-1}\hat{\mathbf{q}}^* - \boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*)_{O_p(n^{-1/2})} + O_p(n^{-2}),$$

where $\mathbf{J}_0^{(3)} \equiv \mathrm{E}\left\{\dfrac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0(\partial \boldsymbol{\theta}_0{'})^{<2>}}\right\}$; $\hat{\mathbf{L}}$ is $\mathbf{L}$ using $\hat{\boldsymbol{\theta}}$ in place of $\boldsymbol{\theta}_0$; and

$(\cdot)_{O_p(n^{-1/2})}$ indicates that $(\cdot)$ is of order $O_p(n^{-1/2})$ (see Ogasawara, 2010, Section 2.3).

The expressions $\mathbf{L}$ and $\mathbf{L}^{-1}$ should be $\hat{\mathbf{L}}$ and $\hat{\mathbf{L}}^{-1}$, respectively in the following passages: the sentence after that including (5.8), Equation (5.9) (two times), the paragraph after Theorem 4 (two times) and Equation (7.1).

## References

Firth, D. (1993). Bias reduction of maximum likelihood estimates. *Biometrika, 80*, 27-38.

Ogasawara, H. (2013). Bias adjustment minimizing the asymptotic mean square error. To appear in *Communications in Statistics – Theory and Methods.* DOI:10.1080/03610926.2013.786788. Online published on April 25, 2013.