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Supplement to Ogasawara's papers on "the ADF pivotal statistic", "mean and covariance structure analysis", and "maximal reliability"

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In Sections 1 through 3 of this note, supplements to Ogasawara (2008, 2009a, 2009b) are given, respectively.

1. Supplement to the paper: "Some properties of the pivotal statistic based on the asymptotically distribution-free theory in structural equation modeling"

Since the proof of Theorem 1 in Ogasawara (2008) was somewhat indirect, another proof of the theorem more directly derived is given in this section.

We show only the essential result $\mathrm{acov}_{\mathrm{NT}}(\hat{\theta},\hat{\alpha}_2)=\mathrm{acov}_{\mathrm{NT}}(\hat{\theta},\hat{\alpha}_{\mathrm{NT2}})$ required for the proof of Theorem 1. From (2.3),

$$n\ \mathrm{acov}_{\mathrm{NT}}(s_{ab},m_{cdef})=\sigma_{abcdef}-\sigma_{ab}\sigma_{cdef}-\sum^{4}\sigma_{abc}\sigma_{def}$$

$$=\sum^{15}\sigma_{ab}\sigma_{cd}\sigma_{ef}-\sigma_{ab}\sum^{3}\sigma_{cd}\sigma_{ef}=\sum^{12}\sigma_{ac}\sigma_{bd}\sigma_{ef}. \tag{A.8}$$

Using the above result,

$$n\,\mathrm{acov}_{\mathrm{NT}}(\hat{\theta},\hat{\alpha}_2)=\frac{\partial\theta}{\partial\boldsymbol{\sigma}'}\boldsymbol{\Omega}\frac{\partial\alpha_2}{\partial\boldsymbol{\sigma}}+\frac{\partial\theta}{\partial\boldsymbol{\sigma}'}\boldsymbol{\Omega}_{24}\frac{\partial\alpha_2}{\partial\boldsymbol{\sigma}_4}$$

$$=\frac{\partial\theta}{\partial\boldsymbol{\sigma}'}\boldsymbol{\Omega}_{\mathrm{NT}}\frac{\partial\alpha_2}{\partial\boldsymbol{\sigma}}+\frac{\partial\theta}{\partial\boldsymbol{\sigma}'}n\,\mathrm{acov}_{\mathrm{NT}}(\mathbf{s},\mathbf{m}_4{}')\frac{\partial\alpha_2}{\partial\boldsymbol{\sigma}_4}$$

$$=\sum_{a\ge b}\sum_{g\ge h}\frac{\partial\theta}{\partial\sigma_{ab}}(\sigma_{ag}\sigma_{bh}+\sigma_{ah}\sigma_{bg})\frac{\partial\alpha_2}{\partial\sigma_{gh}} \tag{A.9}$$

$$+\sum_{a\ge b}\sum_{c\ge d\ge e\ge f}\frac{\partial\theta}{\partial\sigma_{ab}}\sum^{12}\sigma_{ac}\sigma_{bd}\sigma_{ef}\frac{\partial\alpha_2}{\partial\sigma_{cdef}}$$

$$= \sum_{a \ge b} \sum_{g \ge h} \frac{\partial \theta}{\partial \sigma_{ab}} (\sigma_{ag}\sigma_{bh} + \sigma_{ah}\sigma_{bg})$$

$$\times \left\{ \frac{\partial \alpha_{\mathrm{NT2}}}{\partial \sigma_{gh}} - \sum_{i \ge j} \sum_{k \ge l} \frac{\partial \theta}{\partial \sigma_{ij}} \frac{\partial (\sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{kj} + \sigma_{ij}\sigma_{kl})}{\partial \sigma_{gh}} \frac{\partial \theta}{\partial \sigma_{kl}} \right\}$$

$$+ \sum_{a \ge b} \sum_{i \ge j} \sum_{k \ge l} \frac{\partial \theta}{\partial \sigma_{ab}} \sum^{12} \sigma_{ai}\sigma_{bj}\sigma_{kl} \frac{\partial \theta}{\partial \sigma_{ij}} \frac{\partial \theta}{\partial \sigma_{kl}}$$

$$= \frac{\partial \theta}{\partial \boldsymbol{\sigma}'} \boldsymbol{\Omega}_{\mathrm{NT}} \frac{\partial \alpha_{\mathrm{NT2}}}{\partial \boldsymbol{\sigma}} - \sum_{a,b,g,h,i,j,k,l=1}^{p} \frac{\partial \theta}{\partial \tilde{\sigma}_{ab}} (\sigma_{ag}\sigma_{bh} + \sigma_{ah}\sigma_{bg})$$

$$\times \frac{\partial \theta}{\partial \tilde{\sigma}_{ij}} \frac{\partial (\tilde{\sigma}_{ik}\tilde{\sigma}_{jl} + \tilde{\sigma}_{il}\tilde{\sigma}_{kj} + \tilde{\sigma}_{ij}\tilde{\sigma}_{kl})}{\partial \tilde{\sigma}_{gh}} \frac{\partial \theta}{\partial \tilde{\sigma}_{kl}}$$

$$+ \sum_{a,b,i,j,k,l=1}^{p} \frac{\partial \theta}{\partial \tilde{\sigma}_{ab}} \frac{\partial \theta}{\partial \tilde{\sigma}_{ij}} \frac{\partial \theta}{\partial \tilde{\sigma}_{kl}} \sum^{12} \sigma_{ai}\sigma_{bj}\sigma_{kl},$$

where $\sum_{a \ge b}$ is $\sum_{p \ge a \ge b \ge 1}$; $\sum_{c \ge d \ge e \ge f}$ is $\sum_{p \ge c \ge d \ge e \ge f \ge 1}$; and $\tilde{\sigma}_{ab}$ denotes that $\tilde{\sigma}_{ab}$ and $\tilde{\sigma}_{ba}$ $(b \ne a)$ are temporarily treated as different variables in differentiation. The second term on the right-hand side of the last equation of (A.9) becomes

$$- \sum_{a,b,g,h=1}^{p} \frac{\partial \theta}{\partial \tilde{\sigma}_{ab}} (\sigma_{ag}\sigma_{bh} + \sigma_{ah}\sigma_{bg})$$

$$\times \left(\sum_{j,l=1}^{p} \frac{\partial \theta}{\partial \tilde{\sigma}_{gj}} \frac{\partial \theta}{\partial \tilde{\sigma}_{hl}} \sigma_{jl} + \sum_{i,k=1}^{p} \frac{\partial \theta}{\partial \tilde{\sigma}_{ig}} \frac{\partial \theta}{\partial \tilde{\sigma}_{kh}} \sigma_{ik} \right.$$

$$+ \sum_{k,j=1}^{p} \frac{\partial \theta}{\partial \tilde{\sigma}_{gj}} \frac{\partial \theta}{\partial \tilde{\sigma}_{kh}} \sigma_{kj} + \sum_{i,l=1}^{p} \frac{\partial \theta}{\partial \tilde{\sigma}_{ih}} \frac{\partial \theta}{\partial \tilde{\sigma}_{gl}} \sigma_{il}$$

$$\left. + \sum_{k,l=1}^{p} \frac{\partial \theta}{\partial \tilde{\sigma}_{gh}} \frac{\partial \theta}{\partial \tilde{\sigma}_{kl}} \sigma_{kl} + \sum_{i,j=1}^{p} \frac{\partial \theta}{\partial \tilde{\sigma}_{ij}} \frac{\partial \theta}{\partial \tilde{\sigma}_{gh}} \sigma_{ij} \right)$$

$$= -\sum_{a,b,g,h,i,j=1}^{p} \frac{\partial\theta}{\partial\tilde{\sigma}_{ab}} \frac{\partial\theta}{\partial\tilde{\sigma}_{gh}} \frac{\partial\theta}{\partial\tilde{\sigma}_{ij}} \sum^{12} \sigma_{ag}\sigma_{bh}\sigma_{ij}, \tag{A.10}$$

where $\partial\theta / \partial\tilde{\sigma}_{ab} = \partial\theta / \partial\tilde{\sigma}_{ba}$ $(a,b = 1,\ldots, p)$ is used. From (A.9) and (A.10), $\mathrm{acov}_{\mathrm{NT}}(\hat{\theta},\hat{\alpha}_2) = \mathrm{acov}_{\mathrm{NT}}(\hat{\theta},\hat{\alpha}_{\mathrm{NT2}})$ follows.

2. Supplement to the paper "Asymptotic expansions in mean and covariance structure analysis"

In this section, supplement to Ogasawara (2009a) is given, where the subscripts for moments e.g., s_{ab}, $\bar{X}_a$, m_{abcd}, μ_a, σ_{abcd} and κ_{abcd} can take values 1,2,…,*p* with possible duplication.

A.1 The asymptotic covariance matrix of sample means and covariances

Let $s_{ab} = \frac{1}{N-1}\sum_{i=1}^{N}(X_{ia} - \bar{X}_a)(X_{ib} - \bar{X}_b)$, $\bar{X}_a = \frac{1}{N}\sum_{i=1}^{N} X_{ia}$,

$\mathrm{E}(X_{ia}) = \mu_a$ $(i = 1,\ldots, N)$, $S_{ab\cdots d} = \sum_{i=1}^{N}(X_{ia} - \mu_a)(X_{ib} - \mu_b)\cdots(X_{id} - \mu_d)$

especially $S_a = \sum_{i=1}^{N}(X_{ia} - \mu_a)$ and

$\sigma_{ab\cdots d} = \mathrm{E}\{(X_{ia} - \mu_a)(X_{ib} - \mu_b)\cdots(X_{id} - \mu_d)\}$. Then,

$s_{ab} = \frac{1}{N-1}S_{ab} - \frac{1}{N^2 - N}S_a S_b$. For completeness we start with elementary results:

$$\mathrm{Cov}(\bar{X}_a, \bar{X}_b) = \frac{\sigma_{ab}}{N} = \frac{\sigma_{ab}}{N-1} - \frac{\sigma_{ab}}{N^2} + O(N^{-3}),$$

$$\mathrm{Cov}(s_{ab}, \bar{X}_c) = \mathrm{E}\left\{\left(\frac{S_{ab}}{N-1} - \frac{S_a S_b}{N^2 - N} - \sigma_{ab}\right)\frac{S_c}{N}\right\}$$

$$= \frac{\mathrm{E}(S_{ab}S_c)}{N^2 - N} - \frac{\mathrm{E}(S_a S_b S_c)}{N^3 - N^2} = \frac{\sigma_{abc}}{N-1} - \frac{\sigma_{abc}}{N^2 - N}$$

$$= \frac{\sigma_{abc}}{N} = \frac{\sigma_{abc}}{N-1} - \frac{\sigma_{abc}}{N^2} + O(N^{-3}).$$

It is known that

$$\mathrm{Cov}(s_{ab}, s_{cd}) = \frac{\sigma_{abcd} - \sigma_{ab}\sigma_{cd}}{N} + \frac{\sigma_{ac}\sigma_{bd} + \sigma_{ad}\sigma_{bc}}{N^2 - N}$$

$$= \frac{\kappa_{abcd}}{N} + \frac{\sigma_{ac}\sigma_{bd} + \sigma_{ad}\sigma_{bc}}{N-1}$$

(see e.g., Kaplan, 1952, Equation (3)), which gives

$$= \frac{\sigma_{abcd} - \sigma_{ab}\sigma_{cd}}{N-1} - \frac{\kappa_{abcd}}{N^2} + O(N^{-3}).$$

From the above results, (3.7) follows.

A.2 The third cumulants of some sample moments

A.2.1 $\mathrm{E}\{(s_{ab} - \sigma_{ab})(s_{cd} - \sigma_{cd})(\bar{X}_e - \mu_e)\}$

$$= \mathrm{E}\{s_{ab}s_{cd}(\bar{X}_e - \mu_e)\} - \sigma_{ab}\mathrm{E}\{(s_{cd} - \sigma_{cd})(\bar{X}_e - \mu_e)\}$$

$$- \sigma_{cd}\mathrm{E}\{(s_{ab} - \sigma_{ab})(\bar{X}_e - \mu_e)\}$$

$$= \frac{\mathrm{E}(S_{ab}S_{cd}S_e)}{N(N-1)^2} - \frac{\mathrm{E}\{(S_a S_b S_{cd} + S_c S_d S_{ab})S_e\}}{N^2(N-1)^2}$$

$$+ \frac{\mathrm{E}(S_a S_b S_c S_d S_e)}{N^3(N-1)^2} - \frac{\sigma_{ab}\sigma_{cde} + \sigma_{cd}\sigma_{abe}}{N}$$

$$= \frac{N\sigma_{abcde} + (N^2 - N)(\sigma_{ab}\sigma_{cde} + \sigma_{cd}\sigma_{abe})}{N(N-1)^2}$$

$$- \frac{2N\sigma_{abcde} + (N^2 - N)\{\sigma_{ab}\sigma_{cde} + \sigma_{cd}\sigma_{abe} + \sum^{3}(\sigma_{abc}\sigma_{de} + \sigma_{cda}\sigma_{be})\}}{N^2(N-1)^2}$$

$$+\frac{N\sigma_{abcde}+(N^2-N)\sum^{{}_5C_2=10}\sigma_{ab}\sigma_{cde}}{N^3(N-1)^2}-\frac{\sigma_{ab}\sigma_{cde}+\sigma_{cd}\sigma_{abe}}{N}$$

$$=\frac{\sigma_{abcde}}{N^2}+\left\{\frac{N-1}{N(N-1)}-\frac{1}{N}\right\}(\sigma_{ab}\sigma_{cde}+\sigma_{cd}\sigma_{abe})$$

$$-\frac{\sum^{3}(\sigma_{abc}\sigma_{de}+\sigma_{cda}\sigma_{be})}{N(N-1)}+O(N^{-3})$$

$$=\frac{\sigma_{abcde}-\sum^{3}(\sigma_{abc}\sigma_{de}+\sigma_{cda}\sigma_{be})}{N^2}+O(N^{-3}).$$

A.2.2 $\mathrm{E}\{(s_{ab}-\sigma_{ab})(\bar{X}_c-\mu_c)(\bar{X}_d-\mu_d)\}$

$$=\mathrm{E}\{s_{ab}(\bar{X}_c-\mu_c)(\bar{X}_d-\mu_d)\}-\frac{\sigma_{ab}\sigma_{cd}}{N}$$

$$=\frac{\mathrm{E}(S_{ab}S_cS_d)}{N^2(N-1)}-\frac{\mathrm{E}(S_aS_bS_cS_d)}{N^3(N-1)}-\frac{\sigma_{ab}\sigma_{cd}}{N}$$

$$=\frac{(N-1)\sigma_{abcd}}{N^2(N-1)}+\frac{N\sigma_{ab}\sigma_{cd}-\sigma_{ab}\sigma_{cd}-\sigma_{ac}\sigma_{bd}-\sigma_{ad}\sigma_{bc}}{N^2}-\frac{\sigma_{ab}\sigma_{cd}}{N}$$

$$=\frac{\sigma_{abcd}-\sigma_{ab}\sigma_{cd}-\sigma_{ac}\sigma_{bd}-\sigma_{ad}\sigma_{bc}}{N^2}=\frac{\kappa_{abcd}}{N^2}.$$

A.2.3 $\mathrm{E}\{(\bar{X}_a-\mu_a)(\bar{X}_b-\mu_b)(\bar{X}_c-\mu_c)\}$

$$=\frac{\mathrm{E}(S_aS_bS_c)}{N^3}=\frac{\sigma_{abc}}{N^2}.$$

From A.2.1, A.2.2 and A.2.3, we have (3.10a) through (3.10c).

A.3 The fourth central moments of some sample moments

A.3.1 $\mathrm{E}\{(s_{ab}-\sigma_{ab})(s_{cd}-\sigma_{cd})(s_{ef}-\sigma_{ef})(\bar{X}_g-\mu_g)\}$

$$=\mathrm{E}\{s_{ab}s_{cd}s_{ef}(\bar{X}_g-\mu_g)\}-\sum^{3}\sigma_{ab}\mathrm{E}\{s_{cd}s_{ef}(\bar{X}_g-\mu_g)\}$$

$$+\sum^{3}\sigma_{ab}\sigma_{cd}\mathrm{E}\{s_{ef}(\bar{X}_g-\mu_g)\}. \tag{A1}$$

(i) The first term on the right-hand side of (A1)

$$=\frac{\mathrm{E}(S_{ab}S_{cd}S_{ef}S_g)}{N(N-1)^3}-\frac{\sum^{3}\mathrm{E}(S_aS_bS_{cd}S_{ef}S_g)}{N^2(N-1)^3}+\frac{\sum^{3}\mathrm{E}(S_aS_bS_cS_dS_{ef}S_g)}{N^3(N-1)^3}$$

$$-\frac{\mathrm{E}(S_aS_bS_cS_dS_eS_fS_g)}{N^4(N-1)^3}$$

$$=\frac{1}{N(N-1)^3}\left\{N\sigma_{abcdefg}+(N^2-N)\sum^{3}(\sigma_{cdefg}\sigma_{ab}+\sigma_{cdef}\sigma_{abg})\right.$$

$$\left.+N(N-1)(N-2)\sum^{3}\sigma_{abg}\sigma_{cd}\sigma_{ef}\right\}$$

$$-\frac{1}{N^2(N-1)^3}\sum^{3}\left\{(N^2-N)(\sigma_{abefg}\sigma_{cd}+\sigma_{abcdg}\sigma_{ef}+\sigma_{ab}\sigma_{cdefg}+\sigma_{ag}\sigma_{bcdef}\right.$$

$$+\sigma_{bg}\sigma_{acdef}+\sigma_{abg}\sigma_{cdef}+\sum^{4}\sigma_{acd}\sigma_{befg}+\sigma_{cdg}\sigma_{abef}+\sigma_{efg}\sigma_{abcd})$$

$$\left.+N(N-1)(N-2)(\sigma_{ab}\sigma_{cdg}\sigma_{ef}+\sigma_{ab}\sigma_{efg}\sigma_{cd}+\sum^{4}\sigma_{acd}\sigma_{bg}\sigma_{ef}+\sigma_{abg}\sigma_{cd}\sigma_{ef})\right\}$$

$$+\frac{1}{N^3(N-1)^3}\sum^{3}N(N-1)(N-2)\left\{\sigma_{ef}\left(\sum^{4}\sigma_{ag}\sigma_{bcd}+\sum^{6}\sigma_{ab}\sigma_{cdg}\right)\right.$$

$$\left.+\sigma_{efg}\sum^{3}\sigma_{ab}\sigma_{cd}+\sum^{4}\sigma_{efa}(\sigma_{bc}\sigma_{dg}+\sigma_{bd}\sigma_{cg}+\sigma_{cd}\sigma_{bg})\right\}+O(N^{-4})$$

$$=\frac{\sigma_{abcdefg}}{N^3}+\frac{\sum^{3}\sigma_{cdefg}\sigma_{ab}}{(N-1)^2}-\frac{1}{N^3}\sum^{3}(\sigma_{abefg}\sigma_{cd}+\sigma_{abcdg}\sigma_{ef}+\sigma_{cdefg}\sigma_{ab}+\sigma_{bcdef}\sigma_{ag}$$

$$+\sigma_{acdef}\sigma_{bg}) + \frac{\sum^{3}\sigma_{cdef}\sigma_{abg}}{(N-1)^2} - \frac{1}{N^3}\sum^{3}(\sigma_{cdef}\sigma_{abg} + \sum^{4}\sigma_{befg}\sigma_{acd}$$

$$+\sigma_{abef}\sigma_{cdg} + \sigma_{abcd}\sigma_{efg}) + \frac{N-2}{(N-1)^2}\sum^{3}\sigma_{abg}\sigma_{cd}\sigma_{ef}$$

$$-\frac{N-2}{N(N-1)^2}\sum^{3}\left\{(\sigma_{cdg}\sigma_{ef} + \sigma_{efg}\sigma_{cd})\sigma_{ab} + \sum^{4}\sigma_{acd}\sigma_{bg}\sigma_{ef} + \sigma_{abg}\sigma_{cd}\,\sigma_{ef}\right\}$$

$$+\frac{1}{N^3}\sum^{3}\left\{\left(\sum^{4}\sigma_{bcd}\sigma_{ag} + \sum^{6}\sigma_{cdg}\sigma_{ab}\right)\sigma_{ef} + \sigma_{efg}\sum^{3}\sigma_{ab}\sigma_{cd}\right.$$

$$\left. + \sum^{4}\sigma_{efa}(\sigma_{bc}\sigma_{dg} + \sigma_{bd}\sigma_{cg} + \sigma_{cd}\,\sigma_{bg})\right\} + O(N^{-4}).$$

(ii) The second term on the right-hand side of (A1)

$$= -\sum^{3}\sigma_{ab}\left[\frac{\mathrm{E}(S_{cd}S_{ef}S_{g})}{N(N-1)^2} - \frac{\mathrm{E}\{(S_cS_dS_{ef} + S_eS_fS_{cd})S_g\}}{N^2(N-1)^2} + \frac{\mathrm{E}(S_cS_dS_eS_fS_g)}{N^3(N-1)^2}\right]$$

$$= -\sum^{3}\sigma_{ab}\left[\frac{N\sigma_{cdefg} + (N^2-N)(\sigma_{cdg}\sigma_{ef} + \sigma_{efg}\sigma_{cd})}{N(N-1)^2}\right.$$

$$-\frac{1}{N^2(N-1)^2}\{2N\sigma_{cdefg} + (N^2-N)(\sigma_{cdg}\sigma_{ef} + \sigma_{efg}\sigma_{cd}$$

$$+\sigma_{efg}\sigma_{cd} + \sigma_{efc}\sigma_{dg} + \sigma_{efd}\sigma_{cg} + \sigma_{cdg}\sigma_{ef} + \sigma_{cde}\sigma_{fg} + \sigma_{cdf}\sigma_{eg})\}$$

$$\left. + \frac{N^2-N}{N^3(N-1)^2}\left(\sum^{4}\sigma_{cde}\sigma_{fg} + \sum^{6}\sigma_{efg}\sigma_{cd}\right)\right] + O(N^{-4})$$

$$= \sum^{3}\sigma_{ab}\left[-\frac{\sigma_{cdefg}}{(N-1)^2} - \frac{\sigma_{cdg}\sigma_{ef} + \sigma_{efg}\sigma_{cd}}{N-1} + \frac{2\sigma_{cdefg}}{N^3}\right.$$

$$+\frac{1}{N(N-1)}(2\sigma_{cdg}\sigma_{ef} + 2\sigma_{efg}\sigma_{cd} + \sigma_{efc}\sigma_{dg} + \sigma_{efd}\sigma_{cg} + \sigma_{cde}\sigma_{fg} + \sigma_{cdf}\sigma_{eg})$$

$$-\frac{1}{N^3}\left(\sum^{4}\sigma_{cde}\sigma_{fg}+\sum^{6}\sigma_{efg}\sigma_{cd}\right)\Bigg]+O(N^{-4}).$$

(iii) The third term on the right-hand side of (A1)

$$=\frac{1}{N}\sum^{3}\sigma_{ab}\sigma_{cd}\sigma_{efg}$$

(iv) The sum of the terms on the right-hand side of (A1)

$$=\frac{\sigma_{abcdefg}}{N^3}-\frac{\sum^{3}(\sigma_{cdefg}\sigma_{ab}+\sigma_{bcdef}\sigma_{ag}+\sigma_{acdef}\sigma_{bg})}{N^3}+\frac{\sum^{3}\sigma_{cdef}\sigma_{abg}}{(N-1)^2}$$

$$-\frac{1}{N^3}\left(3\sum^{3}\sigma_{cdef}\sigma_{abg}+\sum^{3}\sum^{4}\sigma_{befg}\sigma_{acd}\right)+\left(\frac{1}{N-1}-\frac{1}{(N-1)^2}-\frac{N-2}{N(N-1)^2}\times 3\right.$$

$$\left.+\frac{4}{N(N-1)}+\frac{1}{N^3}-\frac{2}{N-1}+\frac{1}{N}\right)^{*}\sum^{3}\sigma_{abg}\sigma_{cd}\sigma_{ef}+\frac{\sum^{3}\sigma_{efg}(\sigma_{ac}\sigma_{bd}+\sigma_{ad}\sigma_{bc})}{N^3}$$

$$+\left(-\frac{N-2}{N(N-1)^2}+\frac{1}{N^3}+\frac{1}{N(N-1)}\right)^{**}\sum^{3}\sum^{4}\sigma_{acd}\sigma_{bg}\sigma_{ef}$$

$$+\frac{1}{N^3}\sum^{3}\sum^{4}\sigma_{efa}(\sigma_{bc}\sigma_{dg}+\sigma_{bd}\sigma_{cg})+O(N^{-4}),$$

(A2)

where the factors $(\cdot)^{*}$ and $(\cdot)^{**}$ are

$$(\cdot)^{*}=\frac{1}{N(N-1)^2}\{-N(N-1)-N-3(N-2)+4(N-1)$$

$$+1+(N-1)^2\}+O(N^{-4})$$

$$=\frac{-N+4}{N(N-1)^2}+O(N^{-4})=-\frac{1}{(N-1)^2}+\frac{4}{N^3}+O(N^{-4}),$$

$$(\cdot)^{**}=\frac{-(N-2)+1+N-1}{N(N-1)^2}+O(N^{-4})=\frac{2}{N^3}+O(N^{-4}),$$

which gives

$$(A2) = \frac{\sigma_{abcdefg}}{N^3} - \frac{\sum^3 (\sigma_{cdefg}\sigma_{ab} + \sigma_{bcdef}\sigma_{ag} + \sigma_{acdef}\sigma_{bg})}{N^3} + \frac{\sum^3 \sigma_{cdef}\sigma_{abg}}{(N-1)^2}$$

$$- \frac{1}{N^3}\left(3\sum^3 \sigma_{cdef}\sigma_{abg} + \sum^3\sum^4 \sigma_{befg}\sigma_{acd} \right)$$

$$+ \left\{ -\frac{1}{(N-1)^2} + \frac{4}{N^3} \right\} \sum^3 \sigma_{abg}\sigma_{cd}\sigma_{ef}$$

$$+ \frac{\sum^3 \sigma_{efg}(\sigma_{ac}\sigma_{bd} + \sigma_{ad}\sigma_{bc})}{N^3} + \frac{2}{N^3}\sum^3\sum^4 \sigma_{acd}\sigma_{bg}\sigma_{ef}$$

$$+ \frac{1}{N^3}\sum^3\sum^4 \sigma_{efa}(\sigma_{bc}\sigma_{dg} + \sigma_{bd}\sigma_{cg}) + O(N^{-4})$$

$$= \frac{1}{(N-1)^2}\left(\sum^3 \sigma_{cdef}\sigma_{abg} - \sum^3 \sigma_{abg}\sigma_{cd}\sigma_{ef} \right)$$

$$+ \frac{1}{N^3}\left\{ \sigma_{abcdefg} - \sum^3 (\sigma_{cdefg}\sigma_{ab} + \sigma_{bcdef}\sigma_{ag} + \sigma_{acdef}\sigma_{bg}) \right.$$

$$- \left(3\sum^3 \sigma_{cdef}\sigma_{abg} + \sum^3\sum^4 \sigma_{befg}\sigma_{acd} \right) + 4\sum^3 \sigma_{abg}\sigma_{cd}\sigma_{ef}$$

$$+ \sum^3 \sigma_{efg}(\sigma_{ac}\sigma_{bd} + \sigma_{ad}\sigma_{bc}) + 2\sum^3\sum^4 \sigma_{acd}\sigma_{bg}\sigma_{ef} \qquad \text{(A3)}$$

$$\left. + \sum^3\sum^4 \sigma_{efa}(\sigma_{bc}\sigma_{dg} + \sigma_{bd}\sigma_{cg}) \right\} + O(N^{-4}).$$

From (A3), (3.15a) follows.

A.3.2 $\mathrm{E}\{(s_{ab} - \sigma_{ab})(s_{cd} - \sigma_{cd})(\bar{X}_e - \mu_e)(\bar{X}_f - \mu_f)\}$

$$= \mathrm{E}\{s_{ab}s_{cd}(\bar{X}_e - \mu_e)(\bar{X}_f - \mu_f)\} - \sum^2 \sigma_{ab}\mathrm{E}\{s_{cd}(\bar{X}_e - \mu_e)(\bar{X}_f - \mu_f)\}$$

$$+ \sigma_{ab}\sigma_{cd}\mathrm{E}\{(\bar{X}_e - \mu_e)(\bar{X}_f - \mu_f)\}. \qquad \text{(A4)}$$

(i) The first term on the right-hand side of (A4)

$$=\frac{\mathrm{E}(S_{ab}S_{cd}S_eS_f)}{N^2(N-1)^2}-\frac{\sum^{2}\mathrm{E}(S_aS_bS_{cd}S_eS_f)}{N^3(N-1)^2}+\frac{\mathrm{E}(S_aS_bS_cS_dS_eS_f)}{N^4(N-1)^2}$$

$$=\frac{1}{N^2(N-1)^2}\{N\sigma_{abcdef}+(N^2-N)(\sigma_{abcd}\sigma_{ef}+\sigma_{abef}\sigma_{cd}+\sigma_{cdef}\sigma_{ab}$$

$$+\sigma_{abe}\sigma_{cdf}+\sigma_{abf}\sigma_{cde})+N(N-1)(N-2)\sigma_{ab}\sigma_{cd}\sigma_{ef}\}$$

$$-\frac{1}{N^3(N-1)^2}\sum^{2}\{(N^2-N)(\sigma_{abcd}\sigma_{ef}+\sigma_{abef}\sigma_{cd}+\sigma_{cdef}\sigma_{ab}+\sum^{4}\sigma_{acde}\sigma_{bf}$$

$$+\sigma_{acd}\sigma_{bef}+\sigma_{bcd}\sigma_{aef}+\sigma_{cde}\sigma_{abf}+\sigma_{cdf}\sigma_{abe})+N(N-1)(N-2)(\sigma_{ab}\sigma_{ef}$$

$$+\sigma_{ae}\sigma_{bf}+\sigma_{af}\sigma_{be})\sigma_{cd}\}+\frac{1}{N^4(N-1)^2}\sum^{15}N(N-1)(N-2)\sigma_{ab}\sigma_{cd}\sigma_{ef}$$

$$+O(N^{-4})$$

$$=\frac{\sigma_{abcdef}}{N^3}+\left(\frac{1}{N(N-1)}-\frac{2}{N^3}\right)(\sigma_{abcd}\sigma_{ef}+\sigma_{abef}\sigma_{cd}+\sigma_{cdef}\sigma_{ab})$$

$$-\frac{\sum^{8}\sigma_{acde}\sigma_{bf}}{N^3}+\frac{\sigma_{abe}\sigma_{cdf}+\sigma_{abf}\sigma_{cde}}{N(N-1)}-\frac{\sum^{4}\sigma_{acd}\sigma_{bef}}{N^3}-\frac{2}{N^3}(\sigma_{abe}\sigma_{cdf}+\sigma_{abf}\sigma_{cde})$$

$$+\left(\frac{N-2}{N(N-1)}-\frac{2(N-2)}{N^2(N-1)}\right)\sigma_{ab}\sigma_{cd}\sigma_{ef}-\frac{N-2}{N^2(N-1)}\sum^{2}(\sigma_{ae}\sigma_{bf}+\sigma_{af}\sigma_{be})\sigma_{cd}$$

$$+\frac{\sum^{15}\sigma_{ab}\sigma_{cd}\sigma_{ef}}{N^3}+O(N^{-4}).$$

(ii) The second term on the right-hand side of (A4)

$$-\sum^{2}\sigma_{ab}\left\{\frac{\mathrm{E}(S_{cd}S_eS_f)}{N^2(N-1)}-\frac{\mathrm{E}(S_cS_dS_eS_f)}{N^3(N-1)}\right\}$$

$$=-\sum^{2}\sigma_{ab}\left[\frac{N\sigma_{cdef}+(N^2-N)\sigma_{cd}\sigma_{ef}}{N^2(N-1)}\right.$$

$$\left.-\frac{1}{N^3(N-1)}\{N\sigma_{cdef}+(N^2-N)(\sigma_{cd}\sigma_{ef}+\sigma_{ce}\sigma_{df}+\sigma_{cf}\sigma_{de})\}\right]$$

$$=\left(-\frac{1}{N(N-1)}+\frac{1}{N^3}\right)(\sigma_{abef}\sigma_{cd}+\sigma_{cdef}\sigma_{ab})+\left(-\frac{2}{N}+\frac{2}{N^2}\right)\sigma_{ab}\sigma_{cd}\sigma_{ef}$$

$$+\frac{1}{N^2}\sum^{2}\sigma_{ab}(\sigma_{ce}\sigma_{df}+\sigma_{cf}\sigma_{de})+O(N^{-4}).$$

(iii) The third term on the right-hand side of (A4)

$$=\frac{\sigma_{ab}\sigma_{cd}\sigma_{ef}}{N}$$

(iv) The sum of the terms on the right-hand side of (A4)

The coefficient of $\sigma_{ab}\sigma_{cd}\sigma_{ef}$ except that of $(1/N^3)\sum^{15}\sigma_{ab}\sigma_{cd}\sigma_{ef}$ is

$$\left(\frac{1}{N-1}-\frac{4}{N(N-1)}+\frac{4}{N^3}\right)+\left(-\frac{2}{N}+\frac{2}{N^2}\right)+\frac{1}{N}=-\frac{3}{N(N-1)}+\frac{2}{N^2}+\frac{4}{N^3}$$

$$=-\frac{1}{N(N-1)}+\frac{2}{N^3}+O(N^{-4})=-\frac{1}{(N-1)^2}+\frac{3}{N^3}+O(N^{-4}).$$

Using the above result, (A4) becomes

$$=\frac{\sigma_{abcdef}}{N^3}+\left(\frac{1}{N(N-1)}-\frac{2}{N^3}\right)\sigma_{abcd}\sigma_{ef}-\frac{\sigma_{abef}\sigma_{cd}+\sigma_{cdef}\sigma_{ab}}{N^3}$$

$$-\frac{\sum^{8}\sigma_{acde}\sigma_{bf}}{N^3}+\frac{\sigma_{abe}\sigma_{cdf}+\sigma_{abf}\sigma_{cde}}{N(N-1)}-\frac{\sum^{4}\sigma_{acd}\sigma_{bef}}{N^3}$$

$$-\frac{2}{N^3}(\sigma_{abe}\sigma_{cdf}+\sigma_{abf}\sigma_{cde})+\left(-\frac{1}{(N-1)^2}+\frac{3}{N^3}\right)\sigma_{ab}\sigma_{cd}\sigma_{ef}$$

$$+\frac{\sum^{2}\sigma_{ab}(\sigma_{ce}\sigma_{df}+\sigma_{cf}\sigma_{de})}{N^3}+\frac{\sum^{15}\sigma_{ab}\sigma_{cd}\sigma_{ef}}{N^3}+O(N^{-4})$$

$$=\frac{1}{(N-1)^2}(\sigma_{abcd}\sigma_{ef}+\sigma_{abe}\sigma_{cdf}+\sigma_{abf}\sigma_{cde}-\sigma_{ab}\sigma_{cd}\sigma_{ef})$$

$$+\frac{1}{N^3}\Big\{\sigma_{abcdef}-3\sigma_{abcd}\sigma_{ef}-(\sigma_{abef}\sigma_{cd}+\sigma_{cdef}\sigma_{ab})$$

$$-\sum^{8}\sigma_{acde}\sigma_{bf}-3(\sigma_{abe}\sigma_{cdf}+\sigma_{abf}\sigma_{cde})$$

$$-\sum^{4}\sigma_{acd}\sigma_{bef}+3\sigma_{ab}\sigma_{cd}\sigma_{ef}+\sum^{2}\sigma_{ab}(\sigma_{ce}\sigma_{df}+\sigma_{cf}\sigma_{de})$$

$$+\sum^{15}\sigma_{ab}\sigma_{cd}\sigma_{ef}\Big\}+O(N^{-4}),$$

which yields (3.15b).

A.3.3 $\mathrm{E}\{(s_{ab}-\sigma_{ab})(\bar{X}_c-\mu_c)(\bar{X}_d-\mu_d)(\bar{X}_e-\mu_e)\}$

$$=\frac{\mathrm{E}(S_{ab}S_cS_dS_e)}{N^3(N-1)}-\frac{\mathrm{E}(S_aS_bS_cS_dS_e)}{N^4(N-1)}-\frac{\sigma_{ab}\mathrm{E}(S_cS_dS_e)}{N^3}$$

$$=\frac{1}{N^3(N-1)}\{N\sigma_{abcde}+(N^2-N)(\sigma_{cde}\sigma_{ab}+\sum^{3}\sigma_{abc}\sigma_{de})\}$$

$$-\frac{N^2-N}{N^4(N-1)}\sum^{10}\sigma_{abc}\sigma_{de}-\frac{\sigma_{cde}\sigma_{ab}}{N^2}+O(N^{-4}).$$

$$=\frac{\sigma_{abcde}}{N^3}+\frac{\sum^{3}\sigma_{abc}\sigma_{de}}{N^2}-\frac{\sum^{10}\sigma_{abc}\sigma_{de}}{N^3}+O(N^{-4})$$

$$= \frac{\sum^{3} \sigma_{abc}\sigma_{de}}{(N-1)^2} + \frac{1}{N^3}\left(\sigma_{abcde} - 2\sum^{3} \sigma_{abc}\sigma_{de} - \sum^{10} \sigma_{abc}\sigma_{de} \right) + O(N^{-4}).$$

From the above equation, (3.15c) follows.

A.3.4 $\mathrm{E}\{(\bar{X}_a - \mu_a)(\bar{X}_b - \mu_b)(\bar{X}_c - \mu_c)(\bar{X}_d - \mu_d)\}$

$$= \frac{\mathrm{E}(S_a S_b S_c S_d)}{N^4} = \frac{1}{N^4}\left\{ N\sigma_{abcd} + (N^2 - N)\sum^{3} \sigma_{ab}\sigma_{cd} \right\}$$

$$= \frac{\sum^{3} \sigma_{ab}\sigma_{cd}}{(N-1)^2} + \frac{\sigma_{abcd} - 3\sum^{3} \sigma_{ab}\sigma_{cd}}{N^3} + O(N^{-4}),$$

which gives (3.15d).

A.4 The partial derivatives of $\hat{F} = (\bar{\mathbf{x}} - \hat{\boldsymbol{\mu}})'\hat{\boldsymbol{\Sigma}}^{-1}(\bar{\mathbf{x}} - \hat{\boldsymbol{\mu}})$

$$\frac{\partial \hat{F}}{\partial \hat{\theta}_i} = -2\frac{\partial \hat{\boldsymbol{\mu}}'}{\partial \hat{\theta}_i}\hat{\boldsymbol{\Sigma}}^{-1}(\bar{\mathbf{x}} - \hat{\boldsymbol{\mu}}) - (\bar{\mathbf{x}} - \hat{\boldsymbol{\mu}})'\hat{\boldsymbol{\Sigma}}^{-1}\frac{\partial \hat{\boldsymbol{\Sigma}}^{-1}}{\partial \hat{\theta}_i}\hat{\boldsymbol{\Sigma}}^{-1}(\bar{\mathbf{x}} - \hat{\boldsymbol{\mu}})$$

$(i = 1, \ldots, q),$

$$\frac{\partial \hat{F}}{\partial \bar{\mathbf{x}}} = 2\hat{\boldsymbol{\Sigma}}^{-1}(\bar{\mathbf{x}} - \hat{\boldsymbol{\mu}}),$$

$$\frac{\partial^2 \hat{F}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} = -2\frac{\partial^2 \hat{\boldsymbol{\mu}}'}{\partial \hat{\theta}_i \partial \hat{\theta}_j}\hat{\boldsymbol{\Sigma}}^{-1}(\bar{\mathbf{x}} - \hat{\boldsymbol{\mu}}) + 2\sum^{2} \frac{\partial \hat{\boldsymbol{\mu}}'}{\partial \hat{\theta}_i}\hat{\boldsymbol{\Sigma}}^{-1}\frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_j}\hat{\boldsymbol{\Sigma}}^{-1}(\bar{\mathbf{x}} - \hat{\boldsymbol{\mu}})$$

$$+2\frac{\partial \hat{\boldsymbol{\mu}}'}{\partial \hat{\theta}_i}\hat{\boldsymbol{\Sigma}}^{-1}\frac{\partial \hat{\boldsymbol{\mu}}}{\partial \hat{\theta}_j} + 2(\bar{\mathbf{x}} - \hat{\boldsymbol{\mu}})'\hat{\boldsymbol{\Sigma}}^{-1}\frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i}\hat{\boldsymbol{\Sigma}}^{-1}\frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_j}\hat{\boldsymbol{\Sigma}}^{-1}(\bar{\mathbf{x}} - \hat{\boldsymbol{\mu}})$$

$$-(\bar{\mathbf{x}} - \hat{\boldsymbol{\mu}})'\hat{\boldsymbol{\Sigma}}^{-1}\frac{\partial^2 \hat{\boldsymbol{\Sigma}}^{-1}}{\partial \hat{\theta}_i \partial \hat{\theta}_j}\hat{\boldsymbol{\Sigma}}^{-1}(\bar{\mathbf{x}} - \hat{\boldsymbol{\mu}})$$

$(i, j = 1, \ldots, q),$

$$\frac{\partial^2 \hat{F}}{\partial \hat{\theta}_i \partial \overline{\mathbf{x}}} = -2\hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_i} \hat{\mathbf{\Sigma}}^{-1}(\overline{\mathbf{x}} - \hat{\boldsymbol{\mu}}) - 2\hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\boldsymbol{\mu}}}{\partial \hat{\theta}_i} \quad (i = 1, \ldots, q),$$

$$\frac{\partial^3 \hat{F}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} = -2 \frac{\partial^3 \hat{\boldsymbol{\mu}}'}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} \hat{\mathbf{\Sigma}}^{-1}(\overline{\mathbf{x}} - \hat{\boldsymbol{\mu}})$$

$$+2\sum^{3} \frac{\partial^2 \hat{\boldsymbol{\mu}}'}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_k} \hat{\mathbf{\Sigma}}^{-1}(\overline{\mathbf{x}} - \hat{\boldsymbol{\mu}})$$

$$+2\sum^{3} \frac{\partial^2 \hat{\boldsymbol{\mu}}'}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\boldsymbol{\mu}}}{\partial \hat{\theta}_k} + 2\sum^{3} \frac{\partial \hat{\boldsymbol{\mu}}}{\partial \hat{\theta}_i} \hat{\mathbf{\Sigma}}^{-1} \frac{\partial^2 \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_j \partial \hat{\theta}_k} \hat{\mathbf{\Sigma}}^{-1}(\overline{\mathbf{x}} - \hat{\boldsymbol{\mu}})$$

$$-2\sum^{6} \frac{\partial \hat{\boldsymbol{\mu}}'}{\partial \hat{\theta}_i} \hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_j} \hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_k} \hat{\mathbf{\Sigma}}^{-1}(\overline{\mathbf{x}} - \hat{\boldsymbol{\mu}}) - 2\sum^{3} \frac{\partial \hat{\boldsymbol{\mu}}'}{\partial \hat{\theta}_i} \hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_j} \hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\boldsymbol{\mu}}}{\partial \hat{\theta}_k}$$

$$-2\sum^{3} (\overline{\mathbf{x}} - \hat{\boldsymbol{\mu}})' \hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_i} \hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_j} \hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_k} \hat{\mathbf{\Sigma}}^{-1}(\overline{\mathbf{x}} - \hat{\boldsymbol{\mu}})$$

$$+2\sum^{3} (\overline{\mathbf{x}} - \hat{\boldsymbol{\mu}})' \hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_i} \hat{\mathbf{\Sigma}}^{-1} \frac{\partial^2 \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_j \partial \hat{\theta}_k} \hat{\mathbf{\Sigma}}^{-1}(\overline{\mathbf{x}} - \hat{\boldsymbol{\mu}})$$

$$-(\overline{\mathbf{x}} - \hat{\boldsymbol{\mu}})' \hat{\mathbf{\Sigma}}^{-1} \frac{\partial^3 \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} \hat{\mathbf{\Sigma}}^{-1}(\overline{\mathbf{x}} - \hat{\boldsymbol{\mu}}) \quad (i, j, k = 1, \ldots, q),$$

$$\frac{\partial^3 \hat{F}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \overline{\mathbf{x}}} = 2\sum^{2} \hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_i} \hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_j} \hat{\mathbf{\Sigma}}^{-1}(\overline{\mathbf{x}} - \hat{\boldsymbol{\mu}}) + 2\sum^{2} \hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_i} \hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\boldsymbol{\mu}}}{\partial \hat{\theta}_j}$$

$$-2\hat{\mathbf{\Sigma}}^{-1} \frac{\partial^2 \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \hat{\mathbf{\Sigma}}^{-1}(\overline{\mathbf{x}} - \hat{\boldsymbol{\mu}}) - 2\hat{\mathbf{\Sigma}}^{-1} \frac{\partial^2 \hat{\boldsymbol{\mu}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \quad (i, j = 1, \ldots, q),$$

$$\frac{\partial^3 \hat{F}}{\partial \hat{\theta}_i \partial \overline{X}_j \partial \overline{X}_k} = -2\left(\hat{\mathbf{\Sigma}}^{-1} \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_i} \hat{\mathbf{\Sigma}}^{-1} \right)_{jk} \quad (i = 1, \ldots q;\ j, k = 1, \ldots, p),$$

$$\frac{\partial^4 F}{\partial\theta_i\partial\theta_j\partial\theta_k\partial\theta_l} = 2\sum^{4}\frac{\partial^3\boldsymbol{\mu}'}{\partial\theta_i\partial\theta_j\partial\theta_k}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\mu}}{\partial\theta_l}$$

$$-2\sum^{12}\frac{\partial^2\boldsymbol{\mu}'}{\partial\theta_i\partial\theta_j}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\Sigma}}{\partial\theta_k}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\mu}}{\partial\theta_l}$$

$$+2\sum^{3}\frac{\partial^2\boldsymbol{\mu}'}{\partial\theta_i\partial\theta_j}\boldsymbol{\Sigma}^{-1}\frac{\partial^2\boldsymbol{\mu}}{\partial\theta_k\partial\theta_l} - 2\sum^{6}\frac{\partial\boldsymbol{\mu}'}{\partial\theta_i}\boldsymbol{\Sigma}^{-1}\frac{\partial^2\boldsymbol{\Sigma}}{\partial\theta_j\partial\theta_k}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\mu}}{\partial\theta_l}$$

$$+2\sum^{12}\frac{\partial\boldsymbol{\mu}'}{\partial\theta_i}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\Sigma}}{\partial\theta_j}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\Sigma}}{\partial\theta_k}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\mu}}{\partial\theta_l} \quad (i,\, j, k, l = 1,\ldots, q),$$

$$\frac{\partial^4 \hat{F}}{\partial\hat{\theta}_i\partial\hat{\theta}_j\partial\hat{\theta}_k\partial\overline{\mathbf{x}}}\Big|_{\hat{\boldsymbol{\theta}}=\boldsymbol{\theta}} = -2\sum^{6}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\Sigma}}{\partial\theta_i}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\Sigma}}{\partial\theta_j}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\mu}}{\partial\theta_k}$$

$$+2\sum^{3}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\Sigma}}{\partial\theta_i}\boldsymbol{\Sigma}^{-1}\frac{\partial^2\boldsymbol{\mu}}{\partial\theta_j\partial\theta_k}$$

$$+2\sum^{3}\boldsymbol{\Sigma}^{-1}\frac{\partial^2\boldsymbol{\Sigma}}{\partial\theta_i\partial\theta_j}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\mu}}{\partial\theta_k} - 2\boldsymbol{\Sigma}^{-1}\frac{\partial^3\boldsymbol{\mu}}{\partial\theta_i\partial\theta_j\partial\theta_k} \quad (i,\, j, k = 1,\ldots, q),$$

$$\frac{\partial^4 \hat{F}}{\partial\hat{\theta}_i\partial\hat{\theta}_j\partial\overline{X}_k\partial\overline{X}_l}\Big|_{\hat{\boldsymbol{\theta}}=\boldsymbol{\theta}} = 2\left(\sum^{2}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\Sigma}}{\partial\theta_i}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\Sigma}}{\partial\theta_j}\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\frac{\partial^2\boldsymbol{\Sigma}}{\partial\theta_i\partial\theta_j}\boldsymbol{\Sigma}^{-1}\right)_{kl}$$

$$(i,\, j = 1,\ldots q; k, l = 1,\ldots, p).$$

A.5 The covariances of some sample moments

A.5.1 $\mathrm{Cov}(m_{abc}, \overline{X}_e)$

$$=\mathrm{E}\{m_{abc}(\overline{X}_e - \mu_e)\} = \frac{\mathrm{E}(S_{abc}S_e)}{N^2} - \frac{\sum^{3}\mathrm{E}(S_a S_{bc} S_e)}{N^3} + \frac{2\mathrm{E}(S_a S_b S_c S_e)}{N^4}$$

$$= \frac{\sigma_{abce} - \sum^{3}\sigma_{ae}\sigma_{bc}}{N} + O(N^{-2}) = \frac{\kappa_{abce}}{N} + O(N^{-2}).$$

The above result yields (4.7b).

A.5.2 $\mathrm{Cov}(m_{abc}, s_{ef})$

First, we have

$$\mathrm{E}(m_{abc}) = \mathrm{E}\left(\frac{S_{abc}}{N} - \frac{1}{N^2}\sum^{3} S_a S_{bc} + \frac{2S_a S_b S_c}{N^3} \right) = \sigma_{abc} - \frac{3}{N}\sigma_{abc} + \frac{2}{N^2}\sigma_{abc}$$

$$= \frac{(N-1)(N-2)}{N^2}\sigma_{abc}$$

and

$$\mathrm{E}(m_{abc}s_{ef}) = \frac{\mathrm{E}(S_{abc}S_{ef})}{N(N-1)} - \frac{\sum^{3}\mathrm{E}(S_a S_{bc} S_{ef})}{N^2(N-1)} + \frac{2\mathrm{E}(S_a S_b S_c S_{ef})}{N^3(N-1)} - \frac{\mathrm{E}(S_{abc}S_e S_f)}{N^2(N-1)}$$

$$+ \frac{\sum^{3}\mathrm{E}(S_a S_{bc} S_e S_f)}{N^3(N-1)} - \frac{2\mathrm{E}(S_a S_b S_c S_e S_f)}{N^4(N-1)}$$

$$= \frac{N\sigma_{abcef} + (N^2 - N)\sigma_{abc}\sigma_{ef}}{N(N-1)} - \frac{(N^2-N)\left(\sum^{3}\sigma_{aef}\sigma_{bc} + 3\sigma_{abc}\sigma_{ef} \right)}{N^2(N-1)}$$

$$- \frac{(N^2-N)\sigma_{abc}\sigma_{ef}}{N^2(N-1)} + O(N^{-2})$$

$$= \frac{\sigma_{abcef}}{N} + \sigma_{abc}\sigma_{ef} - \frac{\sum^{3}\sigma_{aef}\sigma_{bc}}{N} - \frac{4\sigma_{abc}\sigma_{ef}}{N} + O(N^{-2}).$$

From the above results,

$$\mathrm{Cov}(m_{abc}, s_{ef}) = \frac{1}{N}\left(\sigma_{abcef} - \sigma_{abc}\sigma_{ef} - \sum^{3}\sigma_{aef}\sigma_{bc} \right) + O(N^{-2}),$$

which gives (4.7c).

A.5.3 $\mathrm{Cov}(m_{abcd}, \bar{X}_e)$

$$=\mathrm{E}\{m_{abcd}(\bar{X}_e-\mu_e)\}$$

$$=\frac{\mathrm{E}(S_{abcd}S_e)}{N^2}-\frac{\sum^4\mathrm{E}(S_aS_{bcd}S_e)}{N^3}+\frac{\sum^6\mathrm{E}(S_aS_bS_{cd}S_e)}{N^4}-\frac{3\mathrm{E}(S_aS_bS_cS_dS_e)}{N^5}$$

$$=\frac{\sigma_{abcde}}{N}-\frac{(N^2-N)\sum^4\sigma_{bcd}\sigma_{ae}}{N^3}+O(N^{-2})=\frac{\sigma_{abcde}-\sum^4\sigma_{bcd}\sigma_{ae}}{N}+O(N^{-2}).$$

From the above result, (4.7d) follows.

3. Supplement to the paper: "On the estimators of model-based and maximal reliability"

In this section supplement to Ogasawara (2009b) is given.

S1. The partial derivatives of $\hat{\rho}_i$ (i=1, 2, 4) **and** $\hat{\rho}_{\max 5}$

$$\frac{\partial\hat{\rho}_1}{\partial s_{ab}}=\frac{(2-\delta_{ab})\mathrm{tr}(\hat{\mathbf{\Psi}})}{(\mathbf{1'S1})^2}-\frac{1}{\mathbf{1'S1}}\frac{\partial\,\mathrm{tr}(\hat{\mathbf{\Psi}})}{\partial s_{ab}},$$

$$\frac{\partial^2\hat{\rho}_1}{\partial s_{ab}\partial s_{cd}}=-\frac{2}{(\mathbf{1'S1})^3}(2-\delta_{ab})(2-\delta_{cd})\mathrm{tr}(\hat{\mathbf{\Psi}})$$

$$+\sum_{(ab,cd)}^{2}\frac{2-\delta_{ab}}{(\mathbf{1'S1})^2}\frac{\partial\,\mathrm{tr}(\hat{\mathbf{\Psi}})}{\partial s_{cd}}-\frac{1}{\mathbf{1'S1}}\frac{\partial^2\mathrm{tr}(\hat{\mathbf{\Psi}})}{\partial s_{ab}\partial s_{cd}},$$

$$\frac{\partial^3\hat{\rho}_1}{\partial s_{ab}\partial s_{cd}\partial s_{ef}}=\frac{6}{(\mathbf{1'S1})^4}(2-\delta_{ab})(2-\delta_{cd})(2-\delta_{ef})\mathrm{tr}(\hat{\mathbf{\Psi}})$$

$$+\sum_{(ab,cd,ef)}^{3}\left\{\frac{-2}{(\mathbf{1'S1})^3}(2-\delta_{ab})(2-\delta_{cd})\frac{\partial\,\mathrm{tr}(\hat{\mathbf{\Psi}})}{\partial s_{ef}}+\frac{2-\delta_{ab}}{(\mathbf{1'S1})^2}\frac{\partial^2\mathrm{tr}(\hat{\mathbf{\Psi}})}{\partial s_{cd}\partial s_{ef}}\right\}$$

$$-\frac{1}{\mathbf{1'S1}}\frac{\partial^3\mathrm{tr}(\hat{\mathbf{\Psi}})}{\partial s_{ab}\partial s_{cd}\partial s_{ef}},$$

$$\frac{\partial \hat{\rho}_2}{\partial s_{ab}} = -\frac{(2-\delta_{ab})\mathbf{1}'\hat{\mathbf{\Lambda}}\hat{\mathbf{\Lambda}}'\mathbf{1}}{(\mathbf{1}'\mathbf{S}\mathbf{1})^2} + \frac{2}{\mathbf{1}'\mathbf{S}\mathbf{1}}\mathbf{1}'\hat{\mathbf{\Lambda}}\frac{\partial \hat{\mathbf{\Lambda}}'}{\partial s_{ab}}\mathbf{1},$$

$$\frac{\partial^2 \hat{\rho}_2}{\partial s_{ab}\partial s_{cd}} = \frac{2}{(\mathbf{1}'\mathbf{S}\mathbf{1})^3}(2-\delta_{ab})(2-\delta_{cd})\mathbf{1}'\hat{\mathbf{\Lambda}}\hat{\mathbf{\Lambda}}'\mathbf{1}$$

$$-\sum_{(ab,cd)}^{2}\frac{2(2-\delta_{ab})}{(\mathbf{1}'\mathbf{S}\mathbf{1})^2}\mathbf{1}'\hat{\mathbf{\Lambda}}\frac{\partial \hat{\mathbf{\Lambda}}'}{\partial s_{cd}}\mathbf{1} + \frac{2}{\mathbf{1}'\mathbf{S}\mathbf{1}}\mathbf{1}'\left(\frac{\partial \hat{\mathbf{\Lambda}}}{\partial s_{ab}}\frac{\partial \hat{\mathbf{\Lambda}}'}{\partial s_{cd}} + \hat{\mathbf{\Lambda}}\frac{\partial^2 \hat{\mathbf{\Lambda}}'}{\partial s_{ab}\partial s_{cd}}\right)\mathbf{1},$$

$$\frac{\partial^3 \hat{\rho}_2}{\partial s_{ab}\partial s_{cd}\partial s_{ef}} = -\frac{6}{(\mathbf{1}'\mathbf{S}\mathbf{1})^4}(2-\delta_{ab})(2-\delta_{cd})(2-\delta_{ef})\mathbf{1}'\hat{\mathbf{\Lambda}}\hat{\mathbf{\Lambda}}'\mathbf{1}$$

$$+\sum_{(ab,cd,ef)}^{3}\left\{\frac{4}{(\mathbf{1}'\mathbf{S}\mathbf{1})^3}(2-\delta_{ab})(2-\delta_{cd})\mathbf{1}'\hat{\mathbf{\Lambda}}\frac{\partial \hat{\mathbf{\Lambda}}'}{\partial s_{ef}}\mathbf{1}\right.$$

$$\left.-\frac{2(2-\delta_{ab})}{(\mathbf{1}'\mathbf{S}\mathbf{1})^2}\mathbf{1}'\left(\frac{\partial \hat{\mathbf{\Lambda}}}{\partial s_{cd}}\frac{\partial \hat{\mathbf{\Lambda}}'}{\partial s_{ef}} + \hat{\mathbf{\Lambda}}\frac{\partial^2 \hat{\mathbf{\Lambda}}'}{\partial s_{cd}\partial s_{ef}}\right)\mathbf{1} + \frac{2}{\mathbf{1}'\mathbf{S}\mathbf{1}}\mathbf{1}'\frac{\partial \hat{\mathbf{\Lambda}}}{\partial s_{ab}}\frac{\partial^2 \hat{\mathbf{\Lambda}}'}{\partial s_{cd}\partial s_{ef}}\mathbf{1}\right\}$$

$$+\frac{2}{\mathbf{1}'\mathbf{S}\mathbf{1}}\mathbf{1}'\hat{\mathbf{\Lambda}}\frac{\partial^3 \hat{\mathbf{\Lambda}}'}{\partial s_{ab}\partial s_{cd}\partial s_{ef}}\mathbf{1},$$

$$\frac{\partial \hat{\rho}_4}{\partial s_{ab}} = \frac{\mathrm{tr}(\hat{\mathbf{\Psi}})}{(\mathbf{1}'\hat{\mathbf{\Sigma}}\mathbf{1})^2}\mathbf{1}'\left(2\hat{\mathbf{\Lambda}}\frac{\partial \hat{\mathbf{\Lambda}}'}{\partial s_{ab}} + \frac{\partial \hat{\mathbf{\Psi}}}{\partial s_{ab}}\right)\mathbf{1} - \frac{1}{\mathbf{1}'\hat{\mathbf{\Sigma}}\mathbf{1}}\frac{\partial\, \mathrm{tr}(\hat{\mathbf{\Psi}})}{\partial s_{ab}},$$

$$\frac{\partial^2 \hat{\rho}_4}{\partial s_{ab}\partial s_{cd}} = \frac{-2\mathrm{tr}(\hat{\mathbf{\Psi}})}{(\mathbf{1}'\hat{\mathbf{\Sigma}}\mathbf{1})^3}\mathbf{1}'\left(2\hat{\mathbf{\Lambda}}\frac{\partial \hat{\mathbf{\Lambda}}'}{\partial s_{ab}} + \frac{\partial \hat{\mathbf{\Psi}}}{\partial s_{ab}}\right)\mathbf{1}\mathbf{1}'\left(2\hat{\mathbf{\Lambda}}\frac{\partial \hat{\mathbf{\Lambda}}'}{\partial s_{cd}} + \frac{\partial \hat{\mathbf{\Psi}}}{\partial s_{cd}}\right)\mathbf{1}$$

$$+\sum_{(ab,cd)}^{2}\frac{1}{(\mathbf{1}'\hat{\mathbf{\Sigma}}\mathbf{1})^2}\frac{\partial\, \mathrm{tr}(\hat{\mathbf{\Psi}})}{\partial s_{ab}}\mathbf{1}'\left(2\hat{\mathbf{\Lambda}}\frac{\partial \hat{\mathbf{\Lambda}}'}{\partial s_{cd}} + \frac{\partial \hat{\mathbf{\Psi}}}{\partial s_{cd}}\right)\mathbf{1}$$

$$+\frac{\mathrm{tr}(\hat{\mathbf{\Psi}})}{(\mathbf{1}'\hat{\mathbf{\Sigma}}\mathbf{1})^2}\mathbf{1}'\left(2\frac{\partial \hat{\mathbf{\Lambda}}}{\partial s_{ab}}\frac{\partial \hat{\mathbf{\Lambda}}'}{\partial s_{cd}} + 2\hat{\mathbf{\Lambda}}\frac{\partial^2 \hat{\mathbf{\Lambda}}'}{\partial s_{ab}\partial s_{cd}} + \frac{\partial^2 \hat{\mathbf{\Psi}}}{\partial s_{ab}\partial s_{cd}}\right)\mathbf{1}$$

$$-\frac{1}{\mathbf{1}'\hat{\mathbf{\Sigma}}\mathbf{1}}\frac{\partial^2 \mathrm{tr}(\hat{\mathbf{\Psi}})}{\partial s_{ab}\partial s_{cd}},$$

$$\frac{\partial^3\hat{\rho}_4}{\partial s_{ab}\partial s_{cd}\partial s_{ef}}=\frac{6\,\mathrm{tr}(\hat{\mathbf{\Psi}})}{(\mathbf{1}'\hat{\mathbf{\Sigma}}\mathbf{1})^4}\mathbf{1}'\left(2\hat{\mathbf{\Lambda}}\frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{ab}}+\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{ab}}\right)\mathbf{1}\mathbf{1}'\left(2\hat{\mathbf{\Lambda}}\frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{cd}}+\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{cd}}\right)\mathbf{1}$$

$$\times\mathbf{1}'\left(2\hat{\mathbf{\Lambda}}\frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{ef}}+\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{ef}}\right)\mathbf{1}$$

$$+\sum_{(ab,cd,ef)}^{3}\left\{\frac{-2}{(\mathbf{1}'\hat{\mathbf{\Sigma}}\mathbf{1})^3}\frac{\partial\,\mathrm{tr}(\hat{\mathbf{\Psi}})}{\partial s_{ab}}\mathbf{1}'\left(2\hat{\mathbf{\Lambda}}\frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{cd}}+\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{cd}}\right)\mathbf{1}\mathbf{1}'\left(2\hat{\mathbf{\Lambda}}\frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{ef}}+\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{ef}}\right)\mathbf{1}\right.$$

$$-\frac{2\mathrm{tr}(\hat{\mathbf{\Psi}})}{(\mathbf{1}'\hat{\mathbf{\Sigma}}\mathbf{1})^3}\mathbf{1}'\left(2\hat{\mathbf{\Lambda}}\frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{ab}}+\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{ab}}\right)\mathbf{1}'\left(2\frac{\partial\hat{\mathbf{\Lambda}}}{\partial s_{cd}}\frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{ef}}+2\hat{\mathbf{\Lambda}}\frac{\partial^2\hat{\mathbf{\Lambda}}'}{\partial s_{cd}\partial s_{ef}}+\frac{\partial^2\hat{\mathbf{\Psi}}}{\partial s_{cd}\partial s_{ef}}\right)\mathbf{1}$$

$$+\frac{1}{(\mathbf{1}'\hat{\mathbf{\Sigma}}\mathbf{1})^2}\frac{\partial^2\hat{\mathbf{\Psi}}}{\partial s_{ab}\partial s_{cd}}\mathbf{1}'\left(2\hat{\mathbf{\Lambda}}\frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{ef}}+\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{ef}}\right)\mathbf{1}$$

$$\left.+\frac{1}{(\mathbf{1}'\hat{\mathbf{\Sigma}}\mathbf{1})^2}\frac{\partial\,\mathrm{tr}(\hat{\mathbf{\Psi}})}{\partial s_{ab}}\mathbf{1}'\left(2\frac{\partial\hat{\mathbf{\Lambda}}}{\partial s_{cd}}\frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{ef}}+2\hat{\mathbf{\Lambda}}\frac{\partial^2\hat{\mathbf{\Lambda}}'}{\partial s_{cd}\partial s_{ef}}+\frac{\partial^2\hat{\mathbf{\Psi}}}{\partial s_{cd}\partial s_{ef}}\right)\mathbf{1}\right\}$$

$$+\frac{\mathrm{tr}(\hat{\mathbf{\Psi}})}{(\mathbf{1}'\hat{\mathbf{\Sigma}}\mathbf{1})^2}\mathbf{1}'\left(2\sum_{(ab,cd,ef)}^{3}\frac{\partial\hat{\mathbf{\Lambda}}}{\partial s_{ab}}\frac{\partial^2\hat{\mathbf{\Lambda}}'}{\partial s_{cd}\partial s_{ef}}+2\hat{\mathbf{\Lambda}}\frac{\partial^3\hat{\mathbf{\Lambda}}'}{\partial s_{ab}\partial s_{cd}\partial s_{ef}}+\frac{\partial^3\hat{\mathbf{\Psi}}}{\partial s_{ab}\partial s_{cd}\partial s_{ef}}\right)\mathbf{1}$$

$$-\frac{1}{\mathbf{1}'\hat{\mathbf{\Sigma}}\mathbf{1}}\frac{\partial^3\mathrm{tr}(\hat{\mathbf{\Psi}})}{\partial s_{ab}\partial s_{cd}\partial s_{ef}}$$

$(p\ge a\ge b\ge 1;\ p\ge c\ge d\ge 1;\ p\ge e\ge f\ge 1)$.

For $\hat{\rho}_{\max 5}$, let $\hat{y}_5=\sum_{i=1}^{p}s_{ii}/\hat{\psi}_i$, then $\hat{\rho}_{\max 5}=1-\{1/(\hat{y}_5-p+1)\}$. The partial derivatives of $\hat{y}_5$ with respect to s_{ab}'s are as follows:

$$\frac{\partial\hat{y}_5}{\partial s_{ab}}=\frac{\delta_{ab}}{\hat{\psi}_a}-\sum_{i=1}^{p}\frac{s_{ii}}{\hat{\psi}_i^2}\frac{\partial\hat{\psi}_i}{\partial s_{ab}},$$

$$\frac{\partial^2\hat{y}_5}{\partial s_{ab}\partial s_{cd}}=-\sum_{(ab,cd)}^{2}\frac{\delta_{ab}}{\hat{\psi}_a^2}\frac{\partial\hat{\psi}_a}{\partial s_{cd}}+\sum_{i=1}^{p}\left(\frac{2s_{ii}}{\hat{\psi}_i^3}\frac{\partial\hat{\psi}_i}{\partial s_{ab}}\frac{\partial\hat{\psi}_i}{\partial s_{cd}}-\frac{s_{ii}}{\hat{\psi}_i^2}\frac{\partial^2\hat{\psi}_i}{\partial s_{ab}\partial s_{cd}}\right),$$

$$\frac{\partial^3 \hat{y}_5}{\partial s_{ab}\partial s_{cd}\partial s_{ef}} = \sum_{(ab,cd,ef)}^{3}\left(\frac{2\delta_{ab}}{\hat{\psi}_a^3}\frac{\partial\hat{\psi}_a}{\partial s_{cd}}\frac{\partial\hat{\psi}_a}{\partial s_{ef}} - \frac{\delta_{ab}}{\hat{\psi}_a^2}\frac{\partial^2\hat{\psi}_a}{\partial s_{cd}\partial s_{ef}}\right)$$

$$+\sum_{i=1}^{p}\left(-\frac{6s_{ii}}{\hat{\psi}_i^4}\frac{\partial\hat{\psi}_i}{\partial s_{ab}}\frac{\partial\hat{\psi}_i}{\partial s_{cd}}\frac{\partial\hat{\psi}_i}{\partial s_{ef}} + \sum_{(ab,cd,ef)}^{3}\frac{2s_{ii}}{\hat{\psi}_i^3}\frac{\partial\hat{\psi}_i}{\partial s_{ab}}\frac{\partial^2\hat{\psi}_i}{\partial s_{cd}\partial s_{ef}} - \frac{s_{ii}}{\hat{\psi}_i^2}\frac{\partial^3\hat{\psi}_i}{\partial s_{ab}\partial s_{cd}\partial s_{ef}}\right)$$

$(p \ge a \ge b \ge 1;\ p \ge c \ge d \ge 1;\ p \ge e \ge f \ge 1).$

S2. The partial derivatives of $\hat{\mathbf{H}}$ with respect to s_{ab} 's

(1) $\hat{\mathbf{H}} = \hat{\mathbf{\Psi}}^{-1/2}\mathbf{S}\,\hat{\mathbf{\Psi}}^{-1/2}$ for $\hat{\rho}_{\max 1}$

$$\frac{\partial \hat{h}_{ij}}{\partial s_{ab}} = -\frac{1}{2}\sum_{(i,j)}^{2}\hat{\psi}_i^{-3/2}\hat{\psi}_j^{-1/2}\frac{\partial\hat{\psi}_i}{\partial s_{ab}}s_{ij} + \delta_{ia}\delta_{jb}\hat{\psi}_i^{-1/2}\hat{\psi}_j^{-1/2},$$

$$\frac{\partial^2 \hat{h}_{ij}}{\partial s_{ab}\partial s_{cd}} = \sum_{(i,j)}^{2}\left(\frac{3}{4}\hat{\psi}_i^{-5/2}\hat{\psi}_j^{-1/2}\frac{\partial\hat{\psi}_i}{\partial s_{ab}}\frac{\partial\hat{\psi}_i}{\partial s_{cd}} + \frac{1}{4}\hat{\psi}_i^{-3/2}\hat{\psi}_j^{-3/2}\frac{\partial\hat{\psi}_i}{\partial s_{ab}}\frac{\partial\hat{\psi}_j}{\partial s_{cd}} - \frac{1}{2}\hat{\psi}_i^{-3/2}\hat{\psi}_j^{-1/2}\frac{\partial^2\hat{\psi}_i}{\partial s_{ab}\partial s_{cd}}\right)s_{ij}$$

$$-\sum_{(ab,cd)}^{2}\frac{1}{2}\delta_{ic}\delta_{jd}\left(\hat{\psi}_i^{-3/2}\hat{\psi}_j^{-1/2}\frac{\partial\hat{\psi}_i}{\partial s_{ab}} + \hat{\psi}_i^{-1/2}\hat{\psi}_j^{-3/2}\frac{\partial\hat{\psi}_j}{\partial s_{ab}}\right),$$

$$\frac{\partial^3 \hat{h}_{ij}}{\partial s_{ab}\partial s_{cd}\partial s_{ef}} = \sum_{(i,j)}^{2}\Bigg\{-\frac{15}{8}\hat{\psi}_i^{-7/2}\hat{\psi}_j^{-1/2}\frac{\partial\hat{\psi}_i}{\partial s_{ab}}\frac{\partial\hat{\psi}_i}{\partial s_{cd}}\frac{\partial\hat{\psi}_i}{\partial s_{ef}}$$

$$+\sum_{(ab,cd,ef)}^{3}\left(-\frac{3}{8}\hat{\psi}_i^{-5/2}\hat{\psi}_j^{-3/2}\frac{\partial\hat{\psi}_i}{\partial s_{ab}}\frac{\partial\hat{\psi}_i}{\partial s_{cd}}\frac{\partial\hat{\psi}_j}{\partial s_{ef}} + \frac{3}{4}\hat{\psi}_i^{-5/2}\hat{\psi}_j^{-1/2}\frac{\partial\hat{\psi}_i}{\partial s_{ab}}\frac{\partial^2\hat{\psi}_i}{\partial s_{cd}\partial s_{ef}}\right.$$

$$\left.+\frac{1}{4}\hat{\psi}_i^{-3/2}\hat{\psi}_j^{-3/2}\frac{\partial\hat{\psi}_i}{\partial s_{ab}}\frac{\partial^2\hat{\psi}_j}{\partial s_{cd}\partial s_{ef}}\right) - \frac{1}{2}\hat{\psi}_i^{-3/2}\hat{\psi}_j^{-1/2}\frac{\partial^3\hat{\psi}_i}{\partial s_{ab}\partial s_{cd}\partial s_{ef}}\Bigg\}s_{ij}$$

$$+\sum_{(ab,cd,ef)}^{3}\delta_{ie}\delta_{jf}\sum_{(i,j)}^{2}\left(\frac{3}{4}\hat{\psi}_i^{-5/2}\hat{\psi}_j^{-1/2}\frac{\partial\hat{\psi}_i}{\partial s_{ab}}\frac{\partial\hat{\psi}_i}{\partial s_{cd}}+\frac{1}{4}\hat{\psi}_i^{-3/2}\hat{\psi}_j^{-3/2}\frac{\partial\hat{\psi}_i}{\partial s_{ab}}\frac{\partial\hat{\psi}_j}{\partial s_{cd}}\right.$$

$$\left.-\frac{1}{2}\hat{\psi}_i^{-3/2}\hat{\psi}_j^{-1/2}\frac{\partial^2\hat{\psi}_i}{\partial s_{ab}\partial s_{cd}}\right)$$

$(p \ge i \ge j \ge 1;\, p \ge a \ge b \ge 1;\, p \ge c \ge d \ge 1;\, p \ge e \ge f \ge 1).$

(2) $\hat{\mathbf{H}} = \hat{\mathbf{\Lambda}}'\mathbf{S}^{-1}\hat{\mathbf{\Lambda}}$ for $\hat{\rho}_{\max 2}$

$$\frac{\partial\hat{h}_{ij}}{\partial s_{ab}}=\sum_{(i,j)}^{2}\left\{\left(\frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{ab}}\mathbf{S}^{-1}\hat{\mathbf{\Lambda}}\right)_{ij}-\frac{2-\delta_{ab}}{2}(\mathbf{S}^{-1}\hat{\mathbf{\Lambda}})_{ai}(\mathbf{S}^{-1}\hat{\mathbf{\Lambda}})_{bj}\right\},$$

$$\frac{\partial^2\hat{h}_{ij}}{\partial s_{ab}\partial s_{cd}}=\sum_{(i,j)}^{2}\left[\left(\frac{\partial^2\hat{\mathbf{\Lambda}}'}{\partial s_{ab}\partial s_{cd}}\mathbf{S}^{-1}\hat{\mathbf{\Lambda}}+\frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{ab}}\mathbf{S}^{-1}\frac{\partial\hat{\mathbf{\Lambda}}}{\partial s_{cd}}\right)_{ij}\right.$$

$$\left.-\sum_{(ab,cd)}^{2}\frac{2-\delta_{ab}}{2}\left\{\left(\mathbf{S}^{-1}\frac{\partial\hat{\mathbf{\Lambda}}}{\partial s_{cd}}\right)_{ai}(\mathbf{S}^{-1}\hat{\mathbf{\Lambda}})_{bj}+(\mathbf{S}^{-1}\hat{\mathbf{\Lambda}})_{ai}\left(\mathbf{S}^{-1}\frac{\partial\hat{\mathbf{\Lambda}}}{\partial s_{cd}}\right)_{bj}\right\}\right]$$

$$+\frac{1}{4}(2-\delta_{ab})(2-\delta_{cd})\sum_{(a,b,c,d)}^{8}s^{ac}(\mathbf{S}^{-1}\hat{\mathbf{\Lambda}})_{di}(\mathbf{S}^{-1}\hat{\mathbf{\Lambda}})_{bj},$$

$$\frac{\partial^3\hat{h}_{ij}}{\partial s_{ab}\partial s_{cd}\partial s_{ef}}=\sum_{(i,j)}^{2}\left[\left(\frac{\partial^3\hat{\mathbf{\Lambda}}'}{\partial s_{ab}\partial s_{cd}\partial s_{ef}}\mathbf{S}^{-1}\hat{\mathbf{\Lambda}}+\sum_{(ab,cd,ef)}^{3}\frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{ab}}\mathbf{S}^{-1}\frac{\partial^2\hat{\mathbf{\Lambda}}}{\partial s_{cd}\partial s_{ef}}\right)_{ij}\right.$$

$$-\sum_{(ab,cd,ef)}^{3}\frac{2-\delta_{ab}}{2}\left\{\left(\mathbf{S}^{-1}\frac{\partial\hat{\mathbf{\Lambda}}}{\partial s_{cd}}\right)_{ai}\left(\mathbf{S}^{-1}\frac{\partial\hat{\mathbf{\Lambda}}}{\partial s_{ef}}\right)_{bj}+\left(\mathbf{S}^{-1}\frac{\partial\hat{\mathbf{\Lambda}}}{\partial s_{ef}}\right)_{ai}\left(\mathbf{S}^{-1}\frac{\partial\hat{\mathbf{\Lambda}}}{\partial s_{cd}}\right)_{bj}\right.$$

$$\left.\left.+\left(\mathbf{S}^{-1}\frac{\partial^2\hat{\mathbf{\Lambda}}}{\partial s_{cd}\partial s_{ef}}\right)_{ai}(\mathbf{S}^{-1}\hat{\mathbf{\Lambda}})_{bj}+(\mathbf{S}^{-1}\hat{\mathbf{\Lambda}})_{ai}\left(\mathbf{S}^{-1}\frac{\partial^2\hat{\mathbf{\Lambda}}}{\partial s_{cd}\partial s_{ef}}\right)_{bj}\right\}\right]$$

$$+\sum_{(ab,cd,ef)}^{3}\frac{1}{4}(2-\delta_{ab})(2-\delta_{cd})\sum_{(a,b,c,d,(i,j))}^{16} s^{ac}\left(\mathbf{S}^{-1}\frac{\partial\hat{\mathbf{\Lambda}}}{\partial s_{ef}}\right)_{di}(\mathbf{S}^{-1}\hat{\mathbf{\Lambda}})_{bj}$$

$$-\frac{1}{8}(2-\delta_{ab})(2-\delta_{cd})(2-\delta_{ef})\sum_{(a,b,c,d,e,f)}^{48} s^{ae}s^{fc}(\mathbf{S}^{-1}\hat{\mathbf{\Lambda}})_{di}(\mathbf{S}^{-1}\hat{\mathbf{\Lambda}})_{bj}$$

$(k \ge i \ge j \ge 1;\, p \ge a \ge b \ge 1;\, p \ge c \ge d \ge 1;\, p \ge e \ge f \ge 1),$

where $s^{ab} = (\mathbf{S}^{-1})_{ab}$.

(3) $\hat{\mathbf{H}} = \hat{\mathbf{\Lambda}}'\hat{\mathbf{\Psi}}^{-1}\hat{\mathbf{\Lambda}}$ for $\hat{\rho}_{\max 4}$

$$\frac{\partial \hat{h}_{ij}}{\partial s_{ab}} = \sum_{(i,j)}^{2}\left(\frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{ab}}\hat{\mathbf{\Psi}}^{-1}\hat{\mathbf{\Lambda}}\right)_{ij} - \left(\hat{\mathbf{\Lambda}}'\hat{\mathbf{\Psi}}^{-2}\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{ab}}\hat{\mathbf{\Lambda}}\right)_{ij},$$

$$\frac{\partial^2 \hat{h}_{ij}}{\partial s_{ab}\partial s_{cd}} = \sum_{(i,j)}^{2}\left(\frac{\partial^2\hat{\mathbf{\Lambda}}'}{\partial s_{ab}\partial s_{cd}}\hat{\mathbf{\Psi}}^{-1}\hat{\mathbf{\Lambda}} - \sum_{(ab,cd)}^{2}\frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{ab}}\hat{\mathbf{\Psi}}^{-2}\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{cd}}\hat{\mathbf{\Lambda}} + \frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{ab}}\hat{\mathbf{\Psi}}^{-1}\frac{\partial\hat{\mathbf{\Lambda}}}{\partial s_{cd}}\right)_{ij}$$

$$+\left(2\hat{\mathbf{\Lambda}}'\hat{\mathbf{\Psi}}^{-3}\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{ab}}\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{cd}}\hat{\mathbf{\Lambda}} - \hat{\mathbf{\Lambda}}'\hat{\mathbf{\Psi}}^{-2}\frac{\partial^2\hat{\mathbf{\Psi}}}{\partial s_{ab}\partial s_{cd}}\hat{\mathbf{\Lambda}}\right)_{ij},$$

$$\frac{\partial^3 \hat{h}_{ij}}{\partial s_{ab}\partial s_{cd}\partial s_{ef}} = \sum_{(i,j)}^{2}\left\{\frac{\partial^3\hat{\mathbf{\Lambda}}'}{\partial s_{ab}\partial s_{cd}\partial s_{ef}}\hat{\mathbf{\Psi}}^{-1}\hat{\mathbf{\Lambda}} + \sum_{(ab,cd,ef)}^{3}\left(-\frac{\partial^2\hat{\mathbf{\Lambda}}'}{\partial s_{ab}\partial s_{cd}}\hat{\mathbf{\Psi}}^{-2}\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{ef}}\hat{\mathbf{\Lambda}}\right.\right.$$

$$+\frac{\partial^2\hat{\mathbf{\Lambda}}'}{\partial s_{ab}\partial s_{cd}}\hat{\mathbf{\Psi}}^{-1}\frac{\partial\hat{\mathbf{\Lambda}}}{\partial s_{ef}} - \frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{ab}}\hat{\mathbf{\Psi}}^{-2}\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{cd}}\frac{\partial\hat{\mathbf{\Lambda}}}{\partial s_{ef}} + 2\frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{ab}}\hat{\mathbf{\Psi}}^{-3}\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{cd}}\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{ef}}\hat{\mathbf{\Lambda}}$$

$$\left.\left.-\frac{\partial\hat{\mathbf{\Lambda}}'}{\partial s_{ab}}\hat{\mathbf{\Psi}}^{-2}\frac{\partial^2\hat{\mathbf{\Psi}}}{\partial s_{cd}\partial s_{ef}}\hat{\mathbf{\Lambda}}\right)\right\}_{ij} + \left(\sum_{(ab,cd,ef)}^{3}2\hat{\mathbf{\Lambda}}'\hat{\mathbf{\Psi}}^{-3}\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{ab}}\frac{\partial^2\hat{\mathbf{\Psi}}}{\partial s_{cd}\partial s_{ef}}\hat{\mathbf{\Lambda}}\right.$$

$$\left.-6\hat{\mathbf{\Lambda}}'\hat{\mathbf{\Psi}}^{-4}\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{ab}}\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{cd}}\frac{\partial\hat{\mathbf{\Psi}}}{\partial s_{ef}}\hat{\mathbf{\Lambda}} + \hat{\mathbf{\Lambda}}'\hat{\mathbf{\Psi}}^{-2}\frac{\partial^3\hat{\mathbf{\Psi}}}{\partial s_{ab}\partial s_{cd}\partial s_{ef}}\hat{\mathbf{\Lambda}}\right)_{ij}$$

$(k \ge i \ge j \ge 1;\, p \ge a \ge b \ge 1;\, p \ge c \ge d \ge 1;\, p \ge e \ge f \ge 1).$

S3. The third partial derivatives of the eigenvectors

$$\frac{\partial^3 \hat{u}_{ik}}{\partial \hat{h}_{ab}\partial \hat{h}_{cd}\partial \hat{h}_{ef}} = \sum_{\substack{j=1 \\ j\neq i}}^{p^*}(2-\delta_{ab})\left[\frac{1}{(\hat{\gamma}_i-\hat{\gamma}_j)^2}\right.$$

$$\times\left\{\frac{1}{\hat{\gamma}_i-\hat{\gamma}_j}\frac{\partial(\hat{\gamma}_i-\hat{\gamma}_j)}{\partial \hat{h}_{cd}}\frac{\partial(\hat{\gamma}_i-\hat{\gamma}_j)}{\partial \hat{h}_{ef}}+\frac{1}{2}\frac{\partial^2(-\hat{\gamma}_i+\hat{\gamma}_j)}{\partial \hat{h}_{cd}\partial \hat{h}_{ef}}\right\}$$

$$\times(\hat{u}_{ia}\hat{u}_{jb}+\hat{u}_{ib}\hat{u}_{ja})\hat{u}_{jk}$$

$$+\sum_{(cd,ef)}^{2}\frac{1}{2(\hat{\gamma}_i-\hat{\gamma}_j)^2}\frac{\partial(-\hat{\gamma}_i+\hat{\gamma}_j)}{\partial \hat{h}_{cd}}$$

$$\times\left\{\sum_{(a,b)}^{2}\left(\frac{\partial \hat{u}_{ia}}{\partial \hat{h}_{ef}}\hat{u}_{jb}+\hat{u}_{ia}\frac{\partial \hat{u}_{jb}}{\partial \hat{h}_{ef}}\right)\hat{u}_{jk}+(\hat{u}_{ia}\hat{u}_{jb}+\hat{u}_{ib}\hat{u}_{ja})\frac{\partial \hat{u}_{jk}}{\partial \hat{h}_{ef}}\right\}$$

$$+\frac{1}{2(\hat{\gamma}_i-\hat{\gamma}_j)}\left\{\sum_{(a,b)}^{2}\left\{\left(\frac{\partial^2 \hat{u}_{ia}}{\partial \hat{h}_{cd}\partial \hat{h}_{ef}}\hat{u}_{jb}+\frac{\partial \hat{u}_{ia}}{\partial \hat{h}_{cd}}\frac{\partial \hat{u}_{jb}}{\partial \hat{h}_{ef}}\right.\right.\right.$$

$$\left.+\frac{\partial \hat{u}_{ia}}{\partial \hat{h}_{ef}}\frac{\partial \hat{u}_{jb}}{\partial \hat{h}_{cd}}+\hat{u}_{ia}\frac{\partial^2 \hat{u}_{jb}}{\partial \hat{h}_{cd}\partial \hat{h}_{ef}}\right)\hat{u}_{jk}$$

$$\left.\left.\left.+\sum_{(cd,ef)}^{2}\left(\frac{\partial \hat{u}_{ia}}{\partial \hat{h}_{cd}}\hat{u}_{jb}+\hat{u}_{ia}\frac{\partial \hat{u}_{jb}}{\partial \hat{h}_{cd}}\right)\frac{\partial \hat{u}_{jk}}{\partial \hat{h}_{ef}}\right\}+(\hat{u}_{ia}\hat{u}_{jb}+\hat{u}_{ib}\hat{u}_{ja})\frac{\partial^2 \hat{u}_{jk}}{\partial \hat{h}_{cd}\partial \hat{h}_{ef}}\right\}\right]$$

$$(i,k=1,\ldots p^*;\ p^*\geq a\geq b\geq 1;\ p^*\geq c\geq d\geq 1;\ p^*\geq e\geq f\geq 1).$$

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