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Supplement to the paper "Asymptotic properties of the Bayes and pseudo Bayes estimators of ability in item response theory"

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1. An expository derivation of the inverse expansion of the ability estimator

In this section, an expository derivation of the inverse expansion for $\hat{\theta}_{\mathrm{GW}}$ summarized in (A6) of the appendix of Ogasawara (2013) is given. Note that the three sets of equations for $\boldsymbol{\lambda}_{\mathrm{GW}}^{(k)}{}'\mathbf{l}_{\mathrm{GW}}^{(k)}$ $(k=1,2,3)$ in (A6) are to be used sequentially from lower-order results to the next higher-order ones.

For the first set of equations in (A6), we start with writing (A4) as

$$\hat{\theta}_{\mathrm{GW}}-\theta_0=[-\lambda^{-1}+O_p(n^{-1/2})]\left\{\frac{\partial\bar{l}}{\partial\theta_0}+O_p(n^{-1})\right\}+O_p(n^{-2})$$
$$=-\lambda^{-1}\frac{\partial\bar{l}}{\partial\theta_0}+O_p(n^{-1})\equiv\boldsymbol{\lambda}_{\mathrm{GW}}^{(1)}{}'\mathbf{l}_{\mathrm{GW}}^{(1)}+O_p(n^{-1}), \tag{B1}$$

where note $\partial\bar{l}/\partial\theta_0=O_p(n^{-1/2})$.

For the second set of equations of (A6), write (A4) as

$$\hat{\theta}_{\mathrm{GW}}-\theta_0=[-\lambda^{-1}+\lambda^{-2}m+O_p(n^{-1})]$$
$$\times\left\{\frac{\partial\bar{l}}{\partial\theta_0}+n^{-1}g(\theta_0)+\frac{1}{2}\frac{\partial^3\bar{l}}{\partial\theta_0^3}(\hat{\theta}_{\mathrm{GW}}-\theta_0)^2+O_p(n^{-3/2})\right\}+O_p(n^{-2}), \tag{B2}$$

where

$$\frac{1}{2}\frac{\partial^3 \bar{l}}{\partial\theta_0^3}(\hat{\theta}_{\mathrm{GW}}-\theta_0)^2 = \frac{1}{2}\left\{\mathrm{E_T}\left(\frac{\partial^3 \bar{l}}{\partial\theta_0^3}\right)+O_p(n^{-1/2})\right\}\left(-\lambda^{-1}\frac{\partial \bar{l}}{\partial\theta_0}+O_p(n^{-1})\right)^2$$

$$= \frac{1}{2}\mathrm{E_T}\left(\frac{\partial^3 \bar{l}}{\partial\theta_0^3}\right)\left(-\lambda^{-1}\frac{\partial \bar{l}}{\partial\theta_0}\right)^2 + O_p(n^{-3/2}) \tag{B3}$$

$$= \frac{1}{2}\mathrm{E_T}(j_0^{(3)})\lambda^{-2}\left(\frac{\partial \bar{l}}{\partial\theta_0}\right)^2 + O_p(n^{-3/2}),$$

where (B1) is used with the definition of $j_0^{(3)} \equiv \partial^3 \bar{l} / \partial\theta_0^3$. Noting $\lambda^{-2}m = O_p(n^{-1/2})$ in (B2) and using (B3), (B2) becomes

$$\hat{\theta}_{\mathrm{GW}}-\theta_0 = [-\lambda^{-1}+\lambda^{-2}m]$$

$$\times\left\{\frac{\partial \bar{l}}{\partial\theta_0}+n^{-1}g(\theta_0)+\frac{\lambda^{-2}}{2}\mathrm{E_T}(j_0^{(3)})\left(\frac{\partial \bar{l}}{\partial\theta_0}\right)^2\right\}+O_p(n^{-3/2})$$

$$= -\lambda^{-1}\frac{\partial \bar{l}}{\partial\theta_0}+\left\{\lambda^{-2},-\frac{\lambda^{-3}}{2}\mathrm{E_T}(j_0^{(3)})\right\}\left\{m\frac{\partial \bar{l}}{\partial\theta_0},\left(\frac{\partial \bar{l}}{\partial\theta_0}\right)^2\right\}' - n^{-1}\lambda^{-1}g(\theta_0) \tag{B4}$$

$$+O_p(n^{-3/2}),$$

where $m\,\partial \bar{l} / \partial\theta_0 = O_p(n^{-1})$; $(\partial \bar{l} / \partial\theta_0)^2 = O_p(n^{-1})$; only the term $-n^{-1}\lambda^{-1}g(\theta_0)$ is non-stochastic in the last line; and $\lambda^{-2}m\left\{n^{-1}g(\theta_0)+\frac{\lambda^{-2}}{2}\mathrm{E_T}(j_0^{(3)})\left(\frac{\partial \bar{l}}{\partial\theta_0}\right)^2\right\}$ ($=O_p(n^{-3/2})$) has been absorbed in the residual on the right-hand side of the last equation. Recalling (B1), (B4) is summarized as

$$\hat{\theta}_{\mathrm{GW}}-\theta_0 = \boldsymbol{\lambda}_{\mathrm{GW}}^{(1)}{}'\mathbf{l}_{\mathrm{GW}}^{(1)}+\boldsymbol{\lambda}_{\mathrm{GW}}^{(2)}{}'\mathbf{l}_{\mathrm{GW}}^{(2)}+n^{-1}\eta_{\mathrm{GW}}^{(2)}+O_p(n^{-3/2}), \tag{B5}$$

where

$$\boldsymbol{\lambda}_{\mathrm{GW}}^{(2)}{}'\mathbf{l}_{\mathrm{GW}}^{(2)} \equiv \left\{\lambda^{-2},-\frac{\lambda^{-3}}{2}\mathrm{E_T}(j_0^{(3)})\right\}\left\{m\frac{\partial \bar{l}}{\partial\theta_0},\left(\frac{\partial \bar{l}}{\partial\theta_0}\right)^2\right\}' \quad \text{and}$$

$$\eta_{\mathrm{GW}}^{(2)} \equiv -\lambda^{-1}g(\theta_0). \tag{B6}$$

For the third set of equations in (A6), note that $\frac{1}{2}\frac{\partial^3 \overline{l}}{\partial\theta_0^3}(\hat{\theta}_{\mathrm{GW}}-\theta_0)^2$ in (A4) is written as (use (B4))

$$\begin{aligned}
&\frac{1}{2} j_0^{(3)} \left\{ \boldsymbol{\lambda}_{\mathrm{GW}}^{(1)}{}'\mathbf{l}_{\mathrm{GW}}^{(1)} + \boldsymbol{\lambda}_{\mathrm{GW}}^{(2)}{}'\mathbf{l}_{\mathrm{GW}}^{(2)} + O_p(n^{-3/2}) \right\}^2 \\
&= \frac{1}{2}\left[\mathrm{E_T}(j_0^{(3)}) + \{ j_0^{(3)} - \mathrm{E_T}(j_0^{(3)}) \} \right] \\
&\times \left[\lambda^{-2}\left(\frac{\partial\overline{l}}{\partial\theta_0}\right)^2 - 2\lambda^{-1}\frac{\partial\overline{l}}{\partial\theta_0} \right. \\
&\quad \left. \times \left[\left\{ \lambda^{-2}, -\frac{\lambda^{-3}}{2}\mathrm{E_T}(j_0^{(3)}) \right\} \left\{ m\frac{\partial\overline{l}}{\partial\theta_0}, \left(\frac{\partial\overline{l}}{\partial\theta_0}\right)^2 \right\}' - n^{-1}\lambda^{-1} g(\theta_0) \right] \right] \qquad \text{(B7)} \\
&+ O_p(n^{-2}),
\end{aligned}$$

where $\mathrm{E_T}(j_0^{(3)}) = O(1)$, $j_0^{(3)} - \mathrm{E_T}(j_0^{(3)}) = O_p(n^{-1/2})$ and $-2\lambda^{-1}\frac{\partial\overline{l}}{\partial\theta_0}\left[\ \cdot\ \right] = O_p(n^{-3/2})$. Note also in (A4) that

$$\begin{aligned}
\frac{1}{6}\frac{\partial^4\overline{l}}{\partial\theta_0^4}(\hat{\theta}_{\mathrm{GW}}-\theta_0)^3 &= \frac{1}{6}\left\{ \mathrm{E_T}(j_0^{(4)}) + O_p(n^{-1/2}) \right\} \left\{ -\lambda^{-1}\frac{\partial\overline{l}}{\partial\theta_0} + O_p(n^{-1}) \right\}^3 \\
&= -\frac{\lambda^{-3}}{6}\mathrm{E_T}(j_0^{(4)})\left(\frac{\partial\overline{l}}{\partial\theta_0}\right)^3 + O_p(n^{-2}). \qquad \text{(B8)}
\end{aligned}$$

Inserting (B7) and (B8) into (A4), we have

$$
\begin{aligned}
\hat{\theta}_{\mathrm{GW}} - \theta_0 &= [-\lambda^{-1} + \lambda^{-2}m - \{\lambda^{-3}m^2 - n^{-1}g'(\theta_0)\lambda^{-2}\}] \\
&\times\left\{\frac{\partial \bar{l}}{\partial\theta_0} + n^{-1}g(\theta_0) + \frac{1}{2}\left[\mathrm{E_T}(j_0^{(3)}) + \{j_0^{(3)} - \mathrm{E_T}(j_0^{(3)})\}\right]\right. \\
&\times\left[\lambda^{-2}\left(\frac{\partial \bar{l}}{\partial\theta_0}\right)^2 - 2\lambda^{-1}\frac{\partial \bar{l}}{\partial\theta_0}\right. \\
&\times\left.\left[\left\{\lambda^{-2}, -\frac{\lambda^{-3}}{2}\mathrm{E_T}(j_0^{(3)})\right\}\left\{m\frac{\partial \bar{l}}{\partial\theta_0}, \left(\frac{\partial \bar{l}}{\partial\theta_0}\right)^2\right\}' - n^{-1}\lambda^{-1}g(\theta_0)\right]\right] \\
&\left. - \frac{\lambda^{-3}}{6}\mathrm{E_T}(j_0^{(4)})\left(\frac{\partial \bar{l}}{\partial\theta_0}\right)^3\right\} + O_p(n^{-2}).
\end{aligned}
\tag{B9}
$$

In (B9), the terms of order $O_p(n^{-2/3})$ have stochastic factors $m^2\frac{\partial \bar{l}}{\partial\theta_0}$, $m\left(\frac{\partial \bar{l}}{\partial\theta_0}\right)^2$, $\{j_0^{(3)} - \mathrm{E_T}(j_0^{(3)})\}\left(\frac{\partial \bar{l}}{\partial\theta_0}\right)^2$, $\left(\frac{\partial \bar{l}}{\partial\theta_0}\right)^3$, $n^{-1}m$ or $n^{-1}\frac{\partial \bar{l}}{\partial\theta_0}$ with no term of order $O(n^{-2/3})$. Among these, the term with the factor $m\left(\frac{\partial \bar{l}}{\partial\theta_0}\right)^2$ comes from the sum of two terms as

$$
\begin{aligned}
&\lambda^{-2}m\frac{1}{2}\mathrm{E_T}(j_0^{(3)})\lambda^{-2}\left(\frac{\partial \bar{l}}{\partial\theta_0}\right)^2 \\
&+(-\lambda^{-1})\frac{1}{2}\mathrm{E_T}(j_0^{(3)})\left(-2\lambda^{-1}\frac{\partial \bar{l}}{\partial\theta_0}\right)\lambda^{-2}m\frac{\partial \bar{l}}{\partial\theta_0} \\
&= \frac{3}{2}\lambda^{-4}\mathrm{E_T}(j_0^{(3)})m\left(\frac{\partial \bar{l}}{\partial\theta_0}\right)^2.
\end{aligned}
\tag{B10}
$$

The term with the factor $\left(\frac{\partial \bar{l}}{\partial\theta_0}\right)^3$ also comes from two terms as

$$-\lambda^{-1}\frac{1}{2}\mathrm{E_T}(j_0^{(3)})\left(-2\lambda^{-1}\frac{\partial\bar{l}}{\partial\theta_0}\right)\left(-\frac{\lambda^{-3}}{2}\mathrm{E_T}(j_0^{(3)})\right)\left(\frac{\partial\bar{l}}{\partial\theta_0}\right)^2$$

$$+(-\lambda^{-1})\left\{-\frac{\lambda^{-3}}{6}\mathrm{E_T}(j_0^{(4)})\left(\frac{\partial\bar{l}}{\partial\theta_0}\right)^3\right\} \tag{B11}$$

$$=\left[-\frac{\lambda^{-5}}{2}\{\mathrm{E_T}(j_0^{(3)})\}^2+\frac{\lambda^{-4}}{6}\mathrm{E_T}(j_0^{(4)})\right]\left(\frac{\partial\bar{l}}{\partial\theta_0}\right)^3.$$

Similarly, the term with the factor $n^{-1}\dfrac{\partial\bar{l}}{\partial\theta_0}$ is given by two terms as

$$-\{-n^{-1}g'(\theta_0)\lambda^{-2}\}\frac{\partial\bar{l}}{\partial\theta_0}+(-\lambda^{-1})\frac{1}{2}\mathrm{E_T}(j_0^{(3)})\left(-2\lambda^{-1}\frac{\partial\bar{l}}{\partial\theta_0}\right)(-n^{-1}\lambda^{-1}g(\theta_0))$$

$$=n^{-1}\{\lambda^{-2}g'(\theta_0)-\lambda^{-3}\mathrm{E_T}(j_0^{(3)})g(\theta_0)\}\frac{\partial\bar{l}}{\partial\theta_0}. \tag{B12}$$

Each of the remaining terms of order $O_p(n^{-2/3})$ is composed of a single term as

$$-\lambda^{-3}m^2\frac{\partial\bar{l}}{\partial\theta_0},\quad -\frac{\lambda^{-3}}{2}\{j_0^{(3)}-\mathrm{E_T}(j_0^{(3)})\}\left(\frac{\partial\bar{l}}{\partial\theta_0}\right)^2 \quad\text{and}\quad n^{-1}\lambda^{-2}g(\theta_0)m. \tag{B13}$$

Summing (B10) to (B13) in a vector form, we have

$$\equiv\left[-\lambda^{-3},\ \frac{3}{2}\lambda^{-4}\mathrm{E_T}(j_0^{(3)}),\ -\frac{\lambda^{-3}}{2},\ -\frac{\lambda^{-5}}{2}\{\mathrm{E_T}(j_0^{(3)})\}^2+\frac{\lambda^{-4}}{6}\mathrm{E_T}(j_0^{(4)})\right]$$

$$\times\left[m^2\frac{\partial\bar{l}}{\partial\theta_0},\ m\left(\frac{\partial\bar{l}}{\partial\theta_0}\right)^2,\{j_0^{(3)}-\mathrm{E_T}(j_0^{(3)})\}\left(\frac{\partial\bar{l}}{\partial\theta_0}\right)^2,\left(\frac{\partial\bar{l}}{\partial\theta_0}\right)^3\right]' \tag{B14}$$

$$+n^{-1}\lambda^{-2}g(\theta_0)m+n^{-1}\{\lambda^{-2}g'(\theta_0)-\lambda^{-3}\mathrm{E_T}(j_0^{(3)})g(\theta_0)\}(\partial\bar{l}/\partial\theta_0)$$

$$\equiv\boldsymbol{\lambda}^{(3)}{}'\mathbf{l}_0^{(3)}+n^{-1}\lambda^{-2}g(\theta_0)m+n^{-1}\{\lambda^{-2}g'(\theta_0)-\lambda^{-3}\mathrm{E_T}(j_0^{(3)})g(\theta_0)\}(\partial\bar{l}/\partial\theta_0)$$

$$=\{\boldsymbol{\lambda}^{(3)}{}',\ \ \lambda^{-2}g'(\theta_0),\ \ \lambda^{-2}g'(\theta_0)-\lambda^{-3}\mathrm{E_T}(j_0^{(3)})g(\theta_0)\}$$

$$\times\{\mathbf{l}_0^{(3)}{}',\ \ n^{-1}m,\ \ n^{-1}(\partial\bar{l}/\partial\theta_0)\}'$$

$$\equiv\boldsymbol{\lambda}_{\mathrm{GW}}^{(3)}{}'\mathbf{l}_{\mathrm{GW}}^{(3)},$$

where $\boldsymbol{\lambda}^{(3)}{}'\mathbf{l}_0^{(3)}$ is for $\hat{\theta}_{\mathrm{ML}}$ with $g(\theta_0)=0$.

2. Simulated and asymptotic cumulants of z_{GW} and t_{GW}, and RMSEs of $\hat{\theta}_{\mathrm{GW}}$ under model misspecification

Tables B1 and B2 give the simulated and asymptotic cumulants of z_{GW} and t_{GW}, and RMSEs of $\hat{\theta}_{\mathrm{GW}}$ under m.m. when *n*=300. No observations were discarded in the simulations. Under slight m.m., the correlations of P_k and $P_{\mathrm{T}k}$ over items are .916 and .878 for $\theta=-1$ and 2, respectively while under gross m.m. they are .631 and .496.

Reference

Ogasawara, H. (2013). Asymptotic properties of the Bayes and pseudo Bayes estimators of ability in item response theory. *Journal of Multivariate Analysis, 114*, 359-377.

Table B1. Simulated and asymptotic standard errors of z_{GW} and t_{GW} when the IRT model is false

n=300	Slight misspecification						Gross misspecification					
($n^{1/2}$ ASE	θ = -1			θ =2			θ = -1			θ =2		
of $\hat{\theta}_{\mathrm{GW}}$)	(2.625)			(2.368)			(2.430)			(2.211)		
	SD	ASE	HASE	SD	ASE	HASE	SD	ASE	HASE	SD	ASE	HASE
z_{GW} ML	1.013	1	1.015	1.018	1	1.014	1.010	1	1.012	1.017	1	1.012
BM	.964	1	.968	.970	1	.965	.962	1	.964	.970	1	.963
WL	.999	1	1.003	1.007	1	1.003	.997	1	1.000	1.006	1	1.001
JM	.992	1	.995	1.007	1	1.003	.989	1	.992	1.006	1	1.001
t_{GW} ML	.958	.968	.962	.995	.996	.991	.891	.896	.893	.931	.930	.926
BM	.937	.	.938	.977	.	.972	.870	.	.870	.915	.	.909
WL	.952	.	.955	.990	.	.987	.885	.	.887	.927	.	.922
JM	.950	.	.953	.990	.	.986	.883	.	.884	.927	.	.922

Note. *n*=the number of items, ASE=the asymptotic standard error, SD=the standard deviation from simulations, HASE=the higher-order ASE, ML=maximum likelihood, BM=Bayes modal, WL=weighted likelihood, JM=Jeffreys modal. The dots indicate that the values are the same as those by ML.

Table B2. Simulated and asymptotic cumulants of t_{GW} and the RMSEs of $\hat{\theta}_{\mathrm{GW}}$ when the IRT model is false

n=300		Slight misspecification				Gross misspecification			
		θ = -1		θ =2		θ = -1		θ =2	
		Sim.	Th.	Sim	Th.	Sim.	Th.	Sim	Th.
$\alpha_{\mathrm{GW1}}^{(v)}$	ML	.58	.54	.03	.08	.53	.49	.03	.04
	BM	3.19	3.23	-4.65	-4.69	3.19	3.23	-4.70	-4.77
	WL	1.26	1.23	-.97	-.92	1.23	1.19	-.98	-.97
	JM	1.82	1.80	-.88	-.83	1.80	1.77	-.89	-.88
$\alpha_{\mathrm{GW3}}^{(v)}$	ML	3.16	3.23	-1.71	-1.66	2.86	2.83	-1.48	-1.38
	BM	2.99	.	-1.60	.	2.70	.	-1.38	.
	WL	3.12	.	-1.68	.	2.82	.	-1.45	.
	JM	3.11	.	-1.68	.	2.80	.	-1.45	.
$\alpha_{\mathrm{GW4}}^{(v)}$	ML	-2.8	-1.8	-7.6	-10.5	6.5	5.0	-11.5	-5.5
	BM	-1.2	.	-6.3	.	6.9	.	-10.3	.
	WL	-2.1	.	-7.2	.	6.8	.	-11.2	.
	JM	-1.8	.	-7.1	.	6.9	.	-11.1	.
RMSE of $\hat{\theta}_{\mathrm{GW}}$	ML	.154	.154	.139	.139	.142	.142	.130	.129
	BM	.147	.148	.136	.135	.136	.137	.128	.127
	WL	.151	.152	.138	.137	.140	.140	.128	.128
	JM	.150	.151	.138	.137	.139	.139	.128	.128

Note. *n*=the number of items, Sim.=simulated values, Th.=theoretical or asymptotic values, ML=maximum likelihood, BM=Bayes modal, WL=weighted likelihood, JM=Jeffreys modal. The dots indicate that the values are the same as those by ML.