

**Supplement to the paper “Asymptotic expansions of the distributions
of the chi-square statistic based on the asymptotically
distribution-free theory in covariance structures”**

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This note gives asymptotic moments as applications of Ogasawara (2009, Lemma 2) required for the Bartlett correction of Ogasawara (2009, Theorem 2). The results in Subsections 1.2 and 1.3 of this note are not required for the Bartlett correction but shown here for completeness. In this note, σ_{0abcd} is denoted by σ_{abcd} for simplicity of notation with other similar expressions.

The notation \sum^k is synonymous with Σ^k (the sum of k terms with similar patterns). In the following, subscripts $a, b, c, d, e, f, g, h, i, j, k, l, w, x, y$ and $z = 1, \dots, p$.

Errata will be given at the end of this note.

1. Higher-order asymptotic variances

1.1 $E\{(s_{ab} - \sigma_{ab})(s_{cd} - \sigma_{cd})\}$ (see e.g., Kaplan, 1952, Equation (3))

$$\begin{aligned} &= \frac{1}{N} \kappa_{abcd} + \frac{1}{N-1} (\sigma_{ac} \sigma_{bd} + \sigma_{ad} \sigma_{bc}) \\ &= \frac{1}{N} (\sigma_{abcd} - \sigma_{ab} \sigma_{cd} - \sigma_{ac} \sigma_{bd} - \sigma_{ad} \sigma_{bc}) + \frac{1}{N-1} (\sigma_{ac} \sigma_{bd} + \sigma_{ad} \sigma_{bc}) \\ &= \frac{1}{N} (\sigma_{abcd} - \sigma_{ab} \sigma_{cd}) + \frac{1}{N^2} (\sigma_{ac} \sigma_{bd} + \sigma_{ad} \sigma_{bc}) + O(N^{-3}) \\ &= \frac{1}{n} (\sigma_{abcd} - \sigma_{ab} \sigma_{cd}) - \frac{1}{n^2} (\sigma_{abcd} - \sigma_{ab} \sigma_{cd} - \sigma_{ac} \sigma_{bd} - \sigma_{ad} \sigma_{bc}) + O(n^{-3}) \\ &= \frac{1}{n} (\sigma_{abcd} - \sigma_{ab} \sigma_{cd}) - \frac{1}{n^2} \kappa_{abcd} + O(n^{-3}), \end{aligned}$$

where κ_{abcd} denotes the fourth-order multivariate cumulant for variables X_a, X_b, X_c and X_d .

1.2 $E\{(m_{abcd} - \sigma_{abcd})(s_{ef} - \sigma_{ef})\}$

Recall

$$m_{abcd} = \frac{1}{N} S_{abcd} - \frac{1}{N^2} \sum S_a S_{bcd} + \frac{1}{N^3} \sum S_a S_b S_{cd} - \frac{3}{N^4} S_a S_b S_c S_d,$$

$$s_{ef} = \frac{1}{N-1} S_{ef} - \frac{1}{N(N-1)} S_e S_f.$$

Using Ogasawara (2008, Lemmas 1 and 2),

$$\begin{aligned} E\{(m_{abcd} - \sigma_{abcd})(s_{ef} - \sigma_{ef})\} &= E(m_{abcd} s_{ef}) - E(m_{abcd}) \sigma_{ef} \\ &= \left(\frac{1}{N} - \frac{4}{N^2}\right) \sigma_{abcdef} + \left(1 - \frac{5}{N} + \frac{10}{N^2}\right) \sigma_{abcd} \sigma_{ef} + \frac{2}{N^2} \sum \sigma_{abef} \sigma_{cd} \\ &\quad + \frac{1}{N^2} \sum \sigma_{bcde} \sigma_{af} + \left(-\frac{1}{N} + \frac{4}{N^2}\right) \sum \sigma_{aef} \sigma_{bcd} + \left(\frac{2}{N} - \frac{9}{N^2}\right) \left(\sum \sigma_{ab} \sigma_{cd}\right) \sigma_{ef} \\ &\quad - \frac{1}{N^2} \sum (\sigma_{ae} \sigma_{bf} + \sigma_{af} \sigma_{be}) \sigma_{cd} \\ &\quad - \left(1 - \frac{4}{N} + \frac{6}{N^2}\right) \sigma_{abcd} \sigma_{ef} - \left(\frac{2}{N} - \frac{5}{N^2}\right) \left(\sum \sigma_{ab} \sigma_{cd}\right) \sigma_{ef} + O(N^{-3}) \\ &= \frac{1}{N} (\sigma_{abcdef} - \sigma_{abcd} \sigma_{ef} - \sum \sigma_{aef} \sigma_{bcd}) \\ &\quad + \frac{1}{N^2} \{-4\sigma_{abcdef} + 4\sigma_{abcd} \sigma_{ef} + 2\sum \sigma_{abef} \sigma_{cd} + \sum \sigma_{bcde} \sigma_{af} \\ &\quad + 4\sum \sigma_{aef} \sigma_{bcd} - 4\left(\sum \sigma_{ab} \sigma_{cd}\right) \sigma_{ef} - \sum (\sigma_{ae} \sigma_{bf} + \sigma_{af} \sigma_{be}) \sigma_{cd}\} + O(N^{-3}) \\ &= \frac{1}{n} (\sigma_{abcdef} - \sigma_{abcd} \sigma_{ef} - \sum \sigma_{aef} \sigma_{bcd}) \\ &\quad + \frac{1}{n^2} \{-5\sigma_{abcdef} + 5\sigma_{abcd} \sigma_{ef} + 2\sum \sigma_{abef} \sigma_{cd} + \sum \sigma_{bcde} \sigma_{af} \\ &\quad + 5\sum \sigma_{aef} \sigma_{bcd} - 4\left(\sum \sigma_{ab} \sigma_{cd}\right) \sigma_{ef} - \sum (\sigma_{ae} \sigma_{bf} + \sigma_{af} \sigma_{be}) \sigma_{cd}\} + O(n^{-3}). \end{aligned}$$

1.3 $E\{(m_{abcd} - \sigma_{abcd})(m_{efgh} - \sigma_{efgh})\}$

In the following the subscripts $[\cdot]$ _____ are used to identify terms for confirmation of correspondence in equations.

$$\begin{aligned}
E(m_{abcd}m_{efgh}) = & \underbrace{\frac{1}{N}\sigma_{abcdefg}}_{[1]} + \underbrace{\frac{N-1}{N}\sigma_{abcd}\sigma_{efgh}}_{[2AB]} - \frac{1}{N^3} \{ \underbrace{8N\sigma_{abcdefg}}_{[3]} \\
& + \underbrace{8(N^2 - N)\sigma_{abcd}\sigma_{efgh}}_{[4AB]} + \underbrace{(N^2 - N)\sum^4(\sigma_{abcde}\sigma_{fgh} + \sigma_{efgha}\sigma_{bcd})}_{[5AB]} \} \\
& + \frac{1}{N^4} \{ \underbrace{(N^2 - N)2\sum^6(\sigma_{abcdef}\sigma_{gh} + \sigma_{efghab}\sigma_{cd})}_{[6]} + \underbrace{(N^2 - N)3}_{[7]} \\
& \times \underbrace{\sum^4(\sigma_{abcde}\sigma_{fgh} + \sigma_{efgha}\sigma_{bcd})}_{[8]} + \underbrace{12(N^2 - N)\sigma_{abcd}\sigma_{efgh}}_{[8]} \\
& + \underbrace{N(N-1)(N-2)2\sum^3(\sigma_{abcd}\sigma_{ef}\sigma_{gh} + \sigma_{efgh}\sigma_{ab}\sigma_{cd})}_{[9AB]} \} \\
& - \underbrace{\frac{3}{N^5}N(N-1)(N-2)\sum^3(\sigma_{abcd}\sigma_{ef}\sigma_{gh} + \sigma_{efgh}\sigma_{ab}\sigma_{cd})}_{[10]} \\
& + \frac{1}{N^4} \{ \underbrace{N(N-1)\sum^{16}(\sigma_{bcd fgh}\sigma_{ae} + \sigma_{bcde}\sigma_{fgha})}_{[11]} \\
& + \underbrace{N(N-1)16\sigma_{abcd}\sigma_{efgh}}_{[12]} + \underbrace{N(N-1)4\sum^4(\sigma_{abcde}\sigma_{fgh} + \sigma_{efgha}\sigma_{bcd})}_{[13]} \\
& + \underbrace{N(N-1)(N-2)\sum^{16}\sigma_{bcd}\sigma_{fgh}\sigma_{ae}}_{[14AB]} \} \\
& - \frac{1}{N^5}N(N-1)(N-2) \{ \underbrace{8\sum^3(\sigma_{abcd}\sigma_{ef}\sigma_{gh} + \sigma_{efgh}\sigma_{ab}\sigma_{cd})}_{[15]} \\
& + \underbrace{\sum^{48}(\sigma_{bcde}\sigma_{af}\sigma_{gh} + \sigma_{fgha}\sigma_{eb}\sigma_{cd})}_{[16]} + \underbrace{6\sum^{16}\sigma_{bcd}\sigma_{fgh}\sigma_{ae}}_{[17]} \\
& + \underbrace{2\sum^{24}(\sigma_{aef}\sigma_{bcd}\sigma_{gh} + \sigma_{eab}\sigma_{fgh}\sigma_{cd})}_{[18]} \}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{N^6} N(N-1)(N-2)(N-3) \left(\underbrace{[19] 4 \sum^9 \sigma_{ab} \sigma_{cd} \sigma_{ef} \sigma_{gh}}_{\rightarrow} + \underbrace{[20] \sum^{72} \sigma_{ae} \sigma_{bf} \sigma_{cd} \sigma_{gh}}_{\rightarrow} \right) \\
& + O(N^{-3}) \\
& = \underbrace{[2A] \sigma_{abcd} \sigma_{efgh}}_{\rightarrow} + \frac{1}{N} \left\{ \underbrace{[1] \sigma_{abcdefgh}}_{\rightarrow} - \underbrace{[5A] \sum^4 (\sigma_{abcde} \sigma_{fgh} + \sigma_{efgha} \sigma_{bcd})}_{\rightarrow} \right. \\
& \left. - \underbrace{[2B,4A] 9 \sigma_{abcd} \sigma_{efgh}}_{\rightarrow} + \underbrace{[9A] 2 \sum^3 (\sigma_{abcd} \sigma_{ef} \sigma_{gh} + \sigma_{efgh} \sigma_{ab} \sigma_{cd})}_{\rightarrow} + \underbrace{[14A] \sum^{16} \sigma_{bcd} \sigma_{fgh} \sigma_{ae}}_{\rightarrow} \right\} \\
& + \frac{1}{N^2} \left\{ \underbrace{[3] -8 \sigma_{abcdefgh}}_{\rightarrow} + \underbrace{[11] \sum^{16} (\sigma_{bcdfgh} \sigma_{ae} + \sigma_{bcde} \sigma_{fgha})}_{\rightarrow} \right. \\
& \quad \left. + \underbrace{[4B,8,12] 36 \sigma_{abcd} \sigma_{efgh}}_{\rightarrow} + \underbrace{[5B,7,13] 8 \sum^4 (\sigma_{abcde} \sigma_{fgh} + \sigma_{efgha} \sigma_{bcd})}_{\rightarrow} \right. \\
& \quad \left. + \underbrace{[6] 2 \sum^6 (\sigma_{abcdef} \sigma_{gh} + \sigma_{efghab} \sigma_{cd})}_{\rightarrow} - \underbrace{[16] \sum^{48} (\sigma_{bcde} \sigma_{af} \sigma_{gh} + \sigma_{fgha} \sigma_{eb} \sigma_{cd})}_{\rightarrow} \right. \\
& \quad \left. - \underbrace{[9B,10,15] 17 \sum^3 (\sigma_{abcd} \sigma_{ef} \sigma_{gh} + \sigma_{efgh} \sigma_{ab} \sigma_{cd})}_{\rightarrow} - \underbrace{[14B,17] 9 \sum^{16} \sigma_{bcd} \sigma_{fgh} \sigma_{ae}}_{\rightarrow} \right. \\
& \quad \left. - \underbrace{[18] 2 \sum^{24} (\sigma_{aef} \sigma_{bcd} \sigma_{gh} + \sigma_{eab} \sigma_{fgh} \sigma_{cd})}_{\rightarrow} + \underbrace{[19] 4 \sum^9 \sigma_{ab} \sigma_{cd} \sigma_{ef} \sigma_{gh}}_{\rightarrow} \right. \\
& \quad \left. + \underbrace{[20] \sum^{72} \sigma_{ae} \sigma_{bf} \sigma_{cd} \sigma_{gh}}_{\rightarrow} \right\} + O(N^{-3}) \\
& = \sigma_{abcd} \sigma_{efgh} + \frac{1}{n} \left\{ \sigma_{abcdefgh} - \sum^4 (\sigma_{abcde} \sigma_{fgh} + \sigma_{efgha} \sigma_{bcd}) \right. \\
& \quad \left. - 9 \sigma_{abcd} \sigma_{efgh} + 2 \sum^3 (\sigma_{abcd} \sigma_{ef} \sigma_{gh} + \sigma_{efgh} \sigma_{ab} \sigma_{cd}) + \sum^{16} \sigma_{bcd} \sigma_{fgh} \sigma_{ae} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{n^2} \left\{ -9\sigma_{abcdefg} + \sum_{16} (\sigma_{bcd fgh} \sigma_{ae} + \sigma_{bcde} \sigma_{fgha}) \right. \\
& \quad + 45\sigma_{abcd} \sigma_{efgh} + 9 \sum_{4} (\sigma_{abcde} \sigma_{fgh} + \sigma_{efgha} \sigma_{bcd}) \\
& \quad + 2 \sum_{6} (\sigma_{abcdef} \sigma_{gh} + \sigma_{efghab} \sigma_{cd}) - \sum_{48} (\sigma_{bcde} \sigma_{af} \sigma_{gh} + \sigma_{fgha} \sigma_{eb} \sigma_{cd}) \\
& \quad - 19 \sum_{3} (\sigma_{abcd} \sigma_{ef} \sigma_{gh} + \sigma_{efgh} \sigma_{ab} \sigma_{cd}) - 10 \sum_{16} \sigma_{bcd} \sigma_{fgh} \sigma_{ae} \\
& \quad \left. - 2 \sum_{24} (\sigma_{aef} \sigma_{bcd} \sigma_{gh} + \sigma_{eab} \sigma_{fgh} \sigma_{cd}) + 4 \sum_{9} \sigma_{ab} \sigma_{cd} \sigma_{ef} \sigma_{gh} + \sum_{72} \sigma_{ae} \sigma_{bf} \sigma_{cd} \sigma_{gh} \right\} \\
& + O(n^{-3}).
\end{aligned}$$

On the other hand, from Ogasawara (2008, Lemma 1),

$$\begin{aligned}
-\mathbf{E}(m_{abcd})\sigma_{efgh} - \mathbf{E}(m_{efgh})\sigma_{abcd} &= -2 \left(1 - \frac{4}{N} + \frac{6}{N^2} \right) \sigma_{abcd} \sigma_{efgh} \\
& \quad - \left(\frac{2}{N} - \frac{5}{N^2} \right) \sum_{3} (\sigma_{ab} \sigma_{cd} \sigma_{efgh} + \sigma_{ef} \sigma_{gh} \sigma_{abcd}) + O(N^{-3}) \\
&= -2 \left(1 - \frac{4}{n} + \frac{10}{n^2} \right) \sigma_{abcd} \sigma_{efgh} - \left(\frac{2}{n} - \frac{7}{n^2} \right) \sum_{3} (\sigma_{ab} \sigma_{cd} \sigma_{efgh} + \sigma_{ef} \sigma_{gh} \sigma_{abcd}) \\
& \quad + O(n^{-3}).
\end{aligned}$$

From the above results,

$$\begin{aligned}
& \mathbf{E}\{(m_{abcd} - \sigma_{abcd})(m_{efgh} - \sigma_{efgh})\} \\
&= \frac{1}{N} \left\{ \sigma_{abcdefg} - \sum_{4} (\sigma_{abcde} \sigma_{fgh} + \sigma_{efgha} \sigma_{bcd}) - \sigma_{abcd} \sigma_{efgh} + \sum_{16} \sigma_{bcd} \sigma_{fgh} \sigma_{ae} \right\} \\
& + \frac{1}{N^2} \left\{ -8\sigma_{abcdefg} + \sum_{16} (\sigma_{bcd fgh} \sigma_{ae} + \sigma_{bcde} \sigma_{fgha}) \right. \\
& \quad + 24\sigma_{abcd} \sigma_{efgh} + 8 \sum_{4} (\sigma_{abcde} \sigma_{fgh} + \sigma_{efgha} \sigma_{bcd}) \\
& \quad \left. + 2 \sum_{6} (\sigma_{abcdef} \sigma_{gh} + \sigma_{efghab} \sigma_{cd}) - \sum_{48} (\sigma_{bcde} \sigma_{af} \sigma_{gh} + \sigma_{fgha} \sigma_{eb} \sigma_{cd}) \right\}
\end{aligned}$$

$$\begin{aligned}
& -12 \sum^3 (\sigma_{abcd} \sigma_{ef} \sigma_{gh} + \sigma_{efgh} \sigma_{ab} \sigma_{cd}) - 9 \sum^{16} \sigma_{bcd} \sigma_{fgh} \sigma_{ae} \\
& -2 \sum^{24} (\sigma_{aef} \sigma_{bcd} \sigma_{gh} + \sigma_{eab} \sigma_{fgh} \sigma_{cd}) + 4 \sum^9 \sigma_{ab} \sigma_{cd} \sigma_{ef} \sigma_{gh} + \sum^{72} \sigma_{ae} \sigma_{bf} \sigma_{cd} \sigma_{gh} \} \\
& + O(N^{-3}) \\
& = \frac{1}{n} \{ \sigma_{abcdefgh} - \sum^4 (\sigma_{abcde} \sigma_{fgh} + \sigma_{efgha} \sigma_{bcd}) - \sigma_{abcd} \sigma_{efgh} + \sum^{16} \sigma_{bcd} \sigma_{fgh} \sigma_{ae} \} \\
& + \frac{1}{n^2} \{ -9 \sigma_{abcdefgh} + \sum^{16} (\sigma_{bcd fgh} \sigma_{ae} + \sigma_{bcde} \sigma_{fgha}) \\
& \quad + 25 \sigma_{abcd} \sigma_{efgh} + 9 \sum^4 (\sigma_{abcde} \sigma_{fgh} + \sigma_{efgha} \sigma_{bcd}) \\
& \quad + 2 \sum^6 (\sigma_{abcdef} \sigma_{gh} + \sigma_{efghab} \sigma_{cd}) - \sum^{48} (\sigma_{bcde} \sigma_{af} \sigma_{gh} + \sigma_{fgha} \sigma_{eb} \sigma_{cd}) \\
& -12 \sum^3 (\sigma_{abcd} \sigma_{ef} \sigma_{gh} + \sigma_{efgh} \sigma_{ab} \sigma_{cd}) - 10 \sum^{16} \sigma_{bcd} \sigma_{fgh} \sigma_{ae} \\
& -2 \sum^{24} (\sigma_{aef} \sigma_{bcd} \sigma_{gh} + \sigma_{eab} \sigma_{fgh} \sigma_{cd}) + 4 \sum^9 \sigma_{ab} \sigma_{cd} \sigma_{ef} \sigma_{gh} + \sum^{72} \sigma_{ae} \sigma_{bf} \sigma_{cd} \sigma_{gh} \} \\
& + O(n^{-3}).
\end{aligned}$$

2. The third moments

2.1 $E\{(m_{abcd} - \sigma_{abcd})(s_{ef} - \sigma_{ef})(s_{gh} - \sigma_{gh})\}$

Ogasawara (2008, Lemma 5) gave

$$\begin{aligned}
& \mathbb{E}\{(m_{abcd} - \sigma_{abcd})(s_{ef} - \sigma_{ef})(s_{gh} - \sigma_{gh})\} \\
&= \frac{1}{N^2} \left[\sigma_{abcdefg} - (\sigma_{abcdef}\sigma_{gh} + \sigma_{abcdgh}\sigma_{ef}) \right. \\
&\quad - \sum^4 (\sigma_{bcdef}\sigma_{agh} + \sigma_{bcdgh}\sigma_{aef} + \sigma_{aefgh}\sigma_{bcd}) \\
&\quad - \sum^4 \sigma_{abcde}\sigma_{fgh} - 5\sigma_{abcd}\sigma_{efgh} + 6\sigma_{abcd}\sigma_{ef}\sigma_{gh} - \sum^4 (\sigma_{aef}\sigma_{gh} + \sigma_{agh}\sigma_{ef})\sigma_{bcd} \\
&\quad + \sum^4 (\sigma_{ag}\sigma_{efh} + \sigma_{ah}\sigma_{efg} + \sigma_{ae}\sigma_{ghf} + \sigma_{af}\sigma_{ghe})\sigma_{bcd} \\
&\quad + \sum^{4C_2=6} \{(\sigma_{aef}\sigma_{bgh} + \sigma_{agh}\sigma_{bef})\sigma_{cd} + (\sigma_{acd}\sigma_{bgh} + \sigma_{agh}\sigma_{bcd})\sigma_{ef} \\
&\quad \left. + (\sigma_{acd}\sigma_{bef} + \sigma_{aef}\sigma_{bcd})\sigma_{gh}\} + 2 \sum^3 \sigma_{ab}\sigma_{cd}(\sigma_{efgh} - \sigma_{ef}\sigma_{gh}) \right] + O(N^{-3}),
\end{aligned}$$

where N can be replaced by n .

$$2.2 \quad \mathbb{E}\{(m_{abcd} - \sigma_{abcd})(m_{efgh} - \sigma_{efgh})(s_{ij} - \sigma_{ij})\}$$

Write,

$$m_{abcd} = \frac{1}{N} S_{abcd} - \frac{1}{N^2} \sum^4 S_a S_{bcd} + \frac{1}{N^3} \sum^6 S_a S_b S_{cd} - \frac{3}{N^4} S_a S_b S_c S_d,$$

$$m_{efgh} = \frac{1}{N} S_{efgh} - \frac{1}{N^2} \sum^4 S_e S_{fgh} + \frac{1}{N^3} \sum^6 S_e S_f S_{gh} - \frac{3}{N^4} S_e S_f S_g S_h,$$

$$s_{ij} = \frac{1}{N-1} S_{ij} - \frac{1}{N(N-1)} S_i S_j.$$

Then, first we have

$$\begin{aligned}
& \mathbb{E}(m_{abcd} m_{efgh} s_{ij}) \\
&= \frac{1}{N(N-1)} \sigma_{abcdefg} + \frac{1}{N} \left(\underbrace{[2-1] \sigma_{abcdefg} \sigma_{ij}}_{\rightarrow} + \underbrace{[2-2] \sum^2 \sigma_{abcdij} \sigma_{efgh}}_{\rightarrow} \right) \\
&\quad + \underbrace{[3AB] \frac{N-2}{N} \sigma_{abcd} \sigma_{efgh} \sigma_{ij}}_{\rightarrow}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{N^2} \left[8\sigma_{abcdefg} \sigma_{ij} + \sum^2 \{ 8\sigma_{abcd} \sigma_{efghij} + \sum^4 (\sigma_{aefgh} \sigma_{bcdij} + \sigma_{aij} \sigma_{bcdefgh} + \sigma_{aefghij} \sigma_{bcd}) \} \right] \\
& -_{[5AB]} \frac{N-2}{N^2} \{ 8\sigma_{abcd} \sigma_{efgh} \sigma_{ij} + \sum^2 \sum^4 (\sigma_{aefgh} \sigma_{bcd} \sigma_{ij} + \sigma_{aij} \sigma_{bcd} \sigma_{efgh}) \} \\
& \xrightarrow{\hspace{10em}} \\
& + \frac{N-2}{N^3} \left[12\sigma_{abcd} \sigma_{efgh} \sigma_{ij} + \sum^2 \sum_{(abcd)}^6 [(\sigma_{abefgh} \sigma_{ij} + \sigma_{abij} \sigma_{efgh}) \sigma_{cd} + \sigma_{ab} (\sigma_{cdefgh} \sigma_{ij} \right. \\
& \quad + \sigma_{cdij} \sigma_{efgh} + \sigma_{cd} \sigma_{efghij}) + \sum_{(ab)}^2 \{ \sigma_{acd} (\sigma_{befgh} \sigma_{ij} + \sigma_{efgh} \sigma_{bij}) \\
& \quad \left. + \sigma_{aefgh} (\sigma_{bcd} \sigma_{ij} + \sigma_{bij} \sigma_{cd}) + \sigma_{aij} (\sigma_{bcd} \sigma_{efgh} + \sigma_{befgh} \sigma_{cd}) \}] \right] \\
& +_{[7AB]} \frac{(N-2)(N-3)}{N^3} \sum^2 \sum_{(abcd)}^3 2\sigma_{ab} \sigma_{cd} \sigma_{efgh} \sigma_{ij} \\
& \xrightarrow{\hspace{10em}} \\
& - \frac{(N-2)(N-3)}{N^4} 3 \sum^2 \sum_{(abcd)}^3 \sigma_{ab} \sigma_{cd} \sigma_{efgh} \sigma_{ij} \\
& + \frac{N-2}{N^3} \{ \sum^{16} \sigma_{ae} (\sigma_{bcdfgh} \sigma_{ij} + \sigma_{bcdij} \sigma_{fgh} + \sigma_{fghij} \sigma_{bcd}) + 16\sigma_{abcd} \sigma_{efgh} \sigma_{ij} \\
& \quad + 4 \sum^2 \sigma_{abcd} \sum^4 \sigma_{eij} \sigma_{fgh} + \sum^{16} (\sigma_{afgh} \sigma_{ebcd} \sigma_{ij} + \sigma_{afgh} \sigma_{eij} \sigma_{bcd} + \sigma_{ebcd} \sigma_{aij} \sigma_{fgh}) \\
& \quad + 4 \sum^2 \sum^4 \sigma_{abcde} \sigma_{fgh} \sigma_{ij} + \sum^{16} \sigma_{aeij} \sigma_{bcd} \sigma_{fgh} \} \\
& +_{[10AB]} \frac{(N-2)(N-3)}{N^3} \sum^{16} \sigma_{ae} \sigma_{bcd} \sigma_{fgh} \sigma_{ij} \\
& \xrightarrow{\hspace{10em}} \\
& - \frac{(N-2)(N-3)}{N^4} \sum^2 \sum^6 \sum^4 [\sigma_{ab} (\sigma_{cde} \sigma_{fgh} \sigma_{ij} + \sigma_{cd} \sigma_{efgh} \sigma_{ij} + \sigma_{cd} \sigma_{eij} \sigma_{fgh}) \\
& \quad + \sum_{(ab)}^2 \{ \sigma_{acd} \sigma_{be} \sigma_{fgh} \sigma_{ij} + \sigma_{ae} (\sigma_{bcd} \sigma_{fgh} \sigma_{ij} + \sigma_{bfgh} \sigma_{cd} \sigma_{ij} + \sigma_{bij} \sigma_{cd} \sigma_{fgh}) \\
& \quad \left. + \sigma_{afgh} \sigma_{be} \sigma_{cd} \sigma_{ij} + \sigma_{aij} \sigma_{be} \sigma_{cd} \sigma_{fgh} \} + \sigma_{abe} \sigma_{cd} \sigma_{fgh} \sigma_{ij} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{(N-2)(N-3)(N-4)}{N^5} \sum^{36} (\sigma_{ab}\sigma_{ef} + \sigma_{ae}\sigma_{bf} + \sigma_{af}\sigma_{be}) \sigma_{cd}\sigma_{gh}\sigma_{ij} \\
& - \frac{1}{N^2} (\sigma_{abcdefgh}\sigma_{ij} + \sum^2 \sigma_{abcdi}\sigma_{efghj}) \xrightarrow{[14AB]} \frac{N-2}{N^2} \sigma_{abcd}\sigma_{efgh}\sigma_{ij} \\
& + \frac{N-2}{N^3} \sum^2 \sum^4 \{ (\sigma_{abcd}\sigma_{efgh} + \sigma_{aefgh}\sigma_{bcd})\sigma_{ij} + \sum^2 \sigma_{ai}(\sigma_{bcd}\sigma_{efghj} + \sigma_{bcdj}\sigma_{efgh}) \\
& \quad + \sigma_{aij}\sigma_{bcd}\sigma_{efgh} \} \\
& - \frac{(N-2)(N-3)}{N^4} \sum^2 \sum^6 (\sigma_{ab}\sigma_{ij} + \sigma_{ai}\sigma_{bj} + \sigma_{aj}\sigma_{bi}) \sigma_{cd}\sigma_{efgh} \\
& - \frac{(N-2)(N-3)}{N^4} \sum^{16} (\sigma_{ae}\sigma_{ij} + \sigma_{ai}\sigma_{ej} + \sigma_{aj}\sigma_{ei}) \sigma_{bcd}\sigma_{fgh} + O(N^{-3}) \\
& \xrightarrow{[3A]} \underline{\sigma_{abcd}\sigma_{efgh}\sigma_{ij}} \\
& + \frac{1}{N} \{ \xrightarrow{[2-1]} \sigma_{abcdefgh}\sigma_{ij} \xrightarrow{[3B,5A,14A]} 11 \sigma_{abcd}\sigma_{efgh}\sigma_{ij} + \xrightarrow{[2-2]} \sum^2 \sigma_{abcdij}\sigma_{efgh} \\
& \quad \xrightarrow{[5A]} \sum^2 \sum^4 (\sigma_{aefgh}\sigma_{bcd}\sigma_{ij} + \sigma_{aij}\sigma_{bcd}\sigma_{efgh}) \\
& \quad + \xrightarrow{[7A]} \sum^2 \sum^3_{(abcd)} 2\sigma_{ab}\sigma_{cd}\sigma_{efgh}\sigma_{ij} + \xrightarrow{[10A]} \sum^{16} \sigma_{ae}\sigma_{bcd}\sigma_{fgh}\sigma_{ij} \} \\
& + \frac{1}{N^2} \left[\sigma_{abcdefghij} - \{ 8\sigma_{abcdefgh}\sigma_{ij} + \sum^2 \{ 8\sigma_{abcd}\sigma_{efghij} + \sum^4 (\sigma_{aefgh}\sigma_{bcdij} + \sigma_{aij}\sigma_{bcdefgh} \right. \\
& \quad \left. + \sigma_{aefghij}\sigma_{bcd}) \} \} + 16\sigma_{abcd}\sigma_{efgh}\sigma_{ij} + 2\sum^2 \sum^4 (\sigma_{aefgh}\sigma_{bcd}\sigma_{ij} + \sigma_{aij}\sigma_{bcd}\sigma_{efgh}) \right. \\
& \quad \left. + 12\sigma_{abcd}\sigma_{efgh}\sigma_{ij} + \sum^2 \sum^6_{(abcd)} \{ (\sigma_{abefgh}\sigma_{ij} + \sigma_{abij}\sigma_{efgh})\sigma_{cd} + \sigma_{ab}(\sigma_{cdefgh}\sigma_{ij} \right. \\
& \quad \left. + \sigma_{cdij}\sigma_{efgh} + \sigma_{cd}\sigma_{efghij}) + \sum^2_{(ab)} \{ \sigma_{acd}(\sigma_{befgh}\sigma_{ij} + \sigma_{efgh}\sigma_{bij}) + \sigma_{aefgh}(\sigma_{bcd}\sigma_{ij} + \sigma_{bij}\sigma_{cd}) \right. \\
& \quad \left. + \sigma_{aij}(\sigma_{bcd}\sigma_{efgh} + \sigma_{befgh}\sigma_{cd}) \} \} - 13\sum^2 \sum^3_{(abcd)} \sigma_{ab}\sigma_{cd}\sigma_{efgh}\sigma_{ij} \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum^{16} \sigma_{ae} (\sigma_{bcd fgh} \sigma_{ij} + \sigma_{bcd ij} \sigma_{fgh} + \sigma_{fgh ij} \sigma_{bcd}) + 16 \sigma_{abcd} \sigma_{efgh} \sigma_{ij} + 4 \sum^2 \sigma_{abcd} \sum^4 \sigma_{eij} \sigma_{fgh} \\
& + \sum^{16} (\sigma_{afgh} \sigma_{ebcd} \sigma_{ij} + \sigma_{afgh} \sigma_{eij} \sigma_{bcd} + \sigma_{ebcd} \sigma_{aij} \sigma_{fgh}) \\
& + 4 \sum^2 \sum^4 \sigma_{abcde} \sigma_{fgh} \sigma_{ij} + \sum^{16} \sigma_{aeij} \sigma_{bcd} \sigma_{fgh} - 5 \sum^{16} \sigma_{ae} \sigma_{bcd} \sigma_{fgh} \sigma_{ij} \\
& - \sum^2 \sum^6 \sum^4 \{ \sigma_{ab} (\sigma_{cde} \sigma_{fgh} \sigma_{ij} + \sigma_{cd} \sigma_{efgh} \sigma_{ij} + \sigma_{cd} \sigma_{eij} \sigma_{fgh}) \\
& + \sum_{(ab)}^2 \{ \sigma_{acd} \sigma_{be} \sigma_{fgh} \sigma_{ij} + \sigma_{ae} (\sigma_{bcd} \sigma_{fgh} \sigma_{ij} + \sigma_{bfgh} \sigma_{cd} \sigma_{ij} + \sigma_{bij} \sigma_{cd} \sigma_{fgh}) \\
& \quad + \sigma_{afgh} \sigma_{be} \sigma_{cd} \sigma_{ij} + \sigma_{aij} \sigma_{be} \sigma_{cd} \sigma_{fgh} \} + \sigma_{abe} \sigma_{cd} \sigma_{fgh} \sigma_{ij} \} \\
& + \sum^{36} (\sigma_{ab} \sigma_{ef} + \sigma_{ae} \sigma_{bf} + \sigma_{af} \sigma_{be}) \sigma_{cd} \sigma_{gh} \sigma_{ij} \\
& - (\sigma_{abcdefgh} \sigma_{ij} + \sum^2 \sigma_{abcdi} \sigma_{efghj}) + 2 \sigma_{abcd} \sigma_{efgh} \sigma_{ij} \\
& + \sum^2 \sum^4 \{ (\sigma_{abcd} \sigma_{efgh} + \sigma_{aefgh} \sigma_{bcd}) \sigma_{ij} + \sum^2 \sigma_{ai} (\sigma_{bcd} \sigma_{efghj} + \sigma_{bcdj} \sigma_{efgh}) \\
& \quad + \sigma_{aij} \sigma_{bcd} \sigma_{efgh} \} \\
& - \sum^2 \sum^6 (\sigma_{ab} \sigma_{ij} + \sigma_{ai} \sigma_{bj} + \sigma_{aj} \sigma_{bi}) \sigma_{cd} \sigma_{efgh} \\
& - \sum^{16} (\sigma_{ae} \sigma_{ij} + \sigma_{ai} \sigma_{ej} + \sigma_{aj} \sigma_{ei}) \sigma_{bcd} \sigma_{fgh} \Big] + O(N^{-3}).
\end{aligned}$$

The final results are given from the above ones and the known ones (Subsection 1.3 of this supplement; Ogasawara, 2008, Lemma 2).

2.3 $E\{(m_{abcd} - \sigma_{abcd})(m_{efgh} - \sigma_{efgh})(m_{ijkl} - \sigma_{ijkl})\}$

Write,

$$m_{abcd} = \frac{1}{N} S_{abcd} - \frac{1}{N^2} \sum S_a S_{bcd} + \frac{1}{N^3} \sum S_a S_b S_{cd} - \frac{3}{N^4} S_a S_b S_c S_d,$$

$$m_{efgh} = \frac{1}{N} S_{efgh} - \frac{1}{N^2} \sum S_e S_{fgh} + \frac{1}{N^3} \sum S_e S_f S_{gh} - \frac{3}{N^4} S_e S_f S_g S_h,$$

$$m_{ijkl} = \frac{1}{N} S_{ijkl} - \frac{1}{N^2} \sum S_i S_{jkl} + \frac{1}{N^3} \sum S_i S_j S_{kl} - \frac{3}{N^4} S_i S_j S_k S_l.$$

Then,

$$\begin{aligned} & E(m_{abcd} m_{efgh} m_{ijkl}) \\ &= \frac{1}{N^2} \sigma_{abcdefghijkl} + \frac{N-1}{N^2} \sum \sigma_{abcdefgh} \sigma_{ijkl} + \frac{(N-1)(N-2)}{N^2} \sigma_{abcd} \sigma_{efgh} \sigma_{ijkl} \\ & - \frac{N-1}{N^3} \sum \{ 12 \sigma_{abcdefgh} \sigma_{ijkl} + \sum (\sigma_{aefgh} \sigma_{bcdijkl} + \sigma_{aijkl} \sigma_{bcdefgh} + \sigma_{aefghijkl} \sigma_{bcd}) \} \\ & - \frac{(N-1)(N-2)}{N^3} (12 \sigma_{abcd} \sigma_{efgh} \sigma_{ijkl} + \sum \sum \sum \sigma_{aefgh} \sigma_{bcd} \sigma_{ijkl}) \\ & + \frac{(N-1)(N-2)}{N^4} \left[18 \sigma_{abcd} \sigma_{befgh} \sigma_{ijkl} + \sum \sum \{ \sigma_{ab} (\sum \sigma_{cdefgh} \sigma_{ijkl} + \sigma_{cd} \sigma_{efghijkl}) \right. \\ & \left. + \sum \sigma_{acd} \sum \sigma_{befgh} \sigma_{ijkl} + \sum \sum \sigma_{aefgh} (\sigma_{bcd} \sigma_{ijkl} + \sigma_{bijkl} \sigma_{cd}) + \sum \sigma_{abefgh} \sigma_{cd} \sigma_{ijkl} \right] \\ & + \frac{(N-1)(N-2)(N-3)}{N^4} \sum \sum 2 \sigma_{ab} \sigma_{cd} \sigma_{efgh} \sigma_{ijkl} \\ & - \frac{3(N-1)(N-2)(N-3)}{N^5} \sum \sum \sigma_{ab} \sigma_{cd} \sigma_{efgh} \sigma_{ijkl} \\ & + \frac{(N-1)(N-2)}{N^4} \sum \sum \{ \sigma_{ae} (\sigma_{bcdfgh} \sigma_{ijkl} + \sigma_{bcdijkl} \sigma_{fgh} + \sigma_{bcd} \sigma_{fghijkl}) \\ & \quad + \sigma_{abcd} (\sigma_{efgh} \sigma_{ijkl} + \sigma_{eijkl} \sigma_{fgh}) + \sigma_{afgh} (\sigma_{bcde} \sigma_{ijkl} + \sigma_{bcd} \sigma_{eijkl}) \\ & \quad + \sigma_{aijkl} (\sigma_{bcde} \sigma_{fgh} + \sigma_{bcd} \sigma_{efgh}) + \sigma_{aeijkl} \sigma_{bcd} \sigma_{fgh} + \sum \sigma_{aebcd} \sigma_{fgh} \sigma_{ijkl} \} \\ & + \frac{(N-1)(N-2)(N-3)}{N^4} \sum \sum \sigma_{ae} \sigma_{bcd} \sigma_{fgh} \sigma_{ijkl} \end{aligned}$$

$$\begin{aligned}
& -\frac{(N-1)(N-2)(N-3)}{N^5} \sum \sum \sum^4 [\sigma_{ab} \{ \sigma_{cd} (\sigma_{efgh} \sigma_{ijkl} + \sigma_{eijkl} \sigma_{fgh}) + \sigma_{cde} \sigma_{fgh} \sigma_{ijkl} \} \\
& \quad + \sum_{(ab)}^2 \{ \sigma_{acd} \sigma_{be} \sigma_{fgh} \sigma_{ijkl} + \sigma_{ae} (\sigma_{bcd} \sigma_{fgh} \sigma_{ijkl} + \sigma_{bfg} \sigma_{cd} \sigma_{ijkl} + \sigma_{bijkl} \sigma_{cd} \sigma_{fgh}) \\
& \quad \quad \quad + \sigma_{afgh} \sigma_{be} \sigma_{cd} \sigma_{ijkl} + \sigma_{aijkl} \sigma_{be} \sigma_{cd} \sigma_{fgh} \} + \sigma_{abe} \sigma_{cd} \sigma_{fgh} \sigma_{ijkl}] \\
& + \frac{(N-1)(N-2)(N-3)(N-4)}{N^6} \sum \sum^{36} (\sigma_{ab} \sigma_{ef} + \sigma_{ae} \sigma_{bf} + \sigma_{af} \sigma_{be}) \sigma_{cd} \sigma_{gh} \sigma_{ijkl} \\
& - \frac{(N-1)(N-2)(N-3)}{N^5} \sum \{ \sigma_{aei} \sigma_{bcd} \sigma_{fgh} \sigma_{jkl} + \sum^3 \sigma_{abcd} \sigma_{ei} \sigma_{fgh} \sigma_{jkl} \\
& \quad \quad \quad + \sum^3 (\sigma_{afgh} \sigma_{jkl} + \sigma_{ajkl} \sigma_{fgh}) \sigma_{bcd} \sigma_{ei} \} \\
& + \frac{(N-1)(N-2)(N-3)(N-4)}{N^6} \sum \sum^{96} (\sigma_{ab} \sigma_{ei} + \sigma_{ae} \sigma_{bi} + \sigma_{ai} \sigma_{be}) \sigma_{cd} \sigma_{fgh} \sigma_{jkl} \\
& + O(N^{-3}) \\
& = \sigma_{abcd} \sigma_{efgh} \sigma_{ijkl} \\
& + \frac{1}{N} \left(\sum^3 \sigma_{abcdefgh} \sigma_{ijkl} - 15 \sigma_{abcd} \sigma_{efgh} \sigma_{ijkl} - \sum^3 \sum^4 \sum^2 \sigma_{aefg} \sigma_{bcd} \sigma_{ijkl} \right. \\
& \quad \quad \quad \left. + 2 \sum^3 \sum^3 \sigma_{ab} \sigma_{cd} \sigma_{efgh} \sigma_{ijkl} + \sum^3 \sum^{16} \sigma_{ae} \sigma_{bcd} \sigma_{fgh} \sigma_{ijkl} \right) \\
& + \frac{1}{N^2} \left[\sigma_{abcdefghijkl} - 13 \sum^3 \sigma_{abcdefgh} \sigma_{ijkl} + 104 \sigma_{abcd} \sigma_{efgh} \sigma_{ijkl} \right. \\
& \quad \quad \quad \left. - \sum^3 \sum^4 (\sigma_{aefgh} \sigma_{bcdijkl} + \sigma_{aijkl} \sigma_{bcdefgh} + \sigma_{aefghijkl} \sigma_{bcd}) + 3 \sum^3 \sum^4 \sum^2 \sigma_{aefgh} \sigma_{bcd} \sigma_{ijkl} \right. \\
& \quad \quad \quad \left. + \sum^3 \sum^6 \{ \sigma_{ab} (\sum^2 \sigma_{cdefgh} \sigma_{ijkl} + \sigma_{cd} \sigma_{efghijkl}) + \sum_{(ab)}^2 \sigma_{acd} \sum^2 \sigma_{befgh} \sigma_{ijkl} \right. \\
& \quad \quad \quad \left. + \sum_{(ab)}^2 \sum^2 \sigma_{aefgh} (\sigma_{bcd} \sigma_{ijkl} + \sigma_{bijkl} \sigma_{cd}) + \sum^2 \sigma_{abefgh} \sigma_{cd} \sigma_{ijkl} \} \right. \\
& \quad \quad \quad \left. - 15 \sum^3 \sum^3 \sigma_{ab} \sigma_{cd} \sigma_{efgh} \sigma_{ijkl} \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum^3 \sum^{16} \{ \sigma_{ae} (\sigma_{bcd fgh} \sigma_{ijkl} + \sigma_{bcd i jkl} \sigma_{fgh} + \sigma_{bcd} \sigma_{fgh i jkl}) + \sigma_{abcd} \sigma_{e i jkl} \sigma_{fgh} \\
& \quad + \sigma_{afgh} (\sigma_{bcde} \sigma_{ijkl} + \sigma_{bcd} \sigma_{e i jkl}) + \sigma_{ai jkl} (\sigma_{bcde} \sigma_{fgh} + \sigma_{bcd} \sigma_{efgh}) \\
& \quad + \sigma_{ae i jkl} \sigma_{bcd} \sigma_{fgh} + \sum^2 \sigma_{ae bcd} \sigma_{fgh} \sigma_{ijkl} \} - 6 \sum^3 \sum^{16} \sigma_{ae} \sigma_{bcd} \sigma_{fgh} \sigma_{ijkl} \\
& - \sum^6 \sum^6 \sum^4 [\sigma_{ab} \{ \sigma_{cd} (\sigma_{efgh} \sigma_{ijkl} + \sigma_{e i jkl} \sigma_{fgh}) + \sigma_{cde} \sigma_{fgh} \sigma_{ijkl} \} \\
& \quad + \sum_{(ab)}^2 \{ \sigma_{acd} \sigma_{be} \sigma_{fgh} \sigma_{ijkl} + \sigma_{ae} (\sigma_{bcd} \sigma_{fgh} \sigma_{ijkl} + \sigma_{b fgh} \sigma_{cd} \sigma_{ijkl} + \sigma_{b i jkl} \sigma_{cd} \sigma_{fgh}) \\
& \quad \quad + \sigma_{afgh} \sigma_{be} \sigma_{cd} \sigma_{ijkl} + \sigma_{ai jkl} \sigma_{be} \sigma_{cd} \sigma_{fgh} \} + \sigma_{abe} \sigma_{cd} \sigma_{fgh} \sigma_{ijkl}] \\
& + \sum^3 \sum^{36} (\sigma_{ab} \sigma_{ef} + \sigma_{ae} \sigma_{bf} + \sigma_{af} \sigma_{be}) \sigma_{cd} \sigma_{gh} \sigma_{ijkl} \\
& - \sum^{64} \{ \sigma_{aei} \sigma_{bcd} \sigma_{fgh} \sigma_{jkl} + \sum^3 \sigma_{abcd} \sigma_{ei} \sigma_{fgh} \sigma_{ikl} \\
& \quad \quad + \sum^3 (\sigma_{afgh} \sigma_{jkl} + \sigma_{ajkl} \sigma_{fgh}) \sigma_{bcd} \sigma_{ei} \} \\
& + \sum^3 \sum^{96} (\sigma_{ab} \sigma_{ei} + \sigma_{ae} \sigma_{bi} + \sigma_{ai} \sigma_{be}) \sigma_{cd} \sigma_{fgh} \sigma_{jkl} \Big] + O(N^{-3}).
\end{aligned}$$

Using the results of Subsection 2.2 of this supplement and Ogasawara (2008, Lemma 1), we have the final results.

3. The fourth moments

From Ogasawara (2009, Equation (2.12)),

$$\begin{aligned}
& E\{(m_{abcd} - \sigma_{abcd})(s_{ef} - \sigma_{ef})(s_{gh} - \sigma_{gh})(s_{ij} - \sigma_{ij})\} \\
& = \sum^3 \text{acov}(m_{abcd}, s_{ef}) \text{acov}(s_{gh}, s_{ij}) + O(N^{-3})
\end{aligned}$$

with $N \text{acov}(m_{abcd}, s_{ef}) = \sigma_{abcdef} - \sigma_{abcd} \sigma_{ef} - \sum^4 \sigma_{aef} \sigma_{bcd}$ and

$$N \text{acov}(s_{gh}, s_{ij}) = \sigma_{ghij} - \sigma_{gh} \sigma_{ij};$$

and

$$\begin{aligned} & \mathbf{E}\{(m_{abcd} - \sigma_{abcd})(m_{efgh} - \sigma_{efgh})(s_{ij} - \sigma_{ij})(s_{kl} - \sigma_{kl})\} \\ &= \text{acov}(m_{abcd}, m_{efgh})\text{acov}(s_{ij}, s_{kl}) + \sum^2 \text{acov}(m_{abcd}, s_{ij})\text{acov}(m_{efgh}, s_{kl}) \\ &+ O(N^{-3}), \end{aligned}$$

with

$$\begin{aligned} N \text{acov}(m_{abcd}, m_{efgh}) &= \sigma_{abcdefgh} - \sum^4 (\sigma_{efgha}\sigma_{bcd} + \sigma_{abcde}\sigma_{fgh}) - \sigma_{abcd}\sigma_{efgh} \\ &+ \sum^{16} \sigma_{bcd}\sigma_{fgh}\sigma_{ae}, \end{aligned}$$

where N can be replaced by n .

The results

$$\begin{aligned} & \mathbf{E}\{(m_{abcd} - \sigma_{abcd})(m_{efgh} - \sigma_{efgh})(m_{ijkl} - \sigma_{ijkl})(s_{wx} - \sigma_{wx})\} \text{ and} \\ & \mathbf{E}\{(m_{abcd} - \sigma_{abcd})(m_{efgh} - \sigma_{efgh})(m_{ijkl} - \sigma_{ijkl})(m_{wxyz} - \sigma_{wxyz})\} \end{aligned}$$

are similarly given.

Errata

The term $-2\omega^{AB}(\mathbf{R}^*)_{CD}$ in Equations (4.6) and (4.7) should be $-\omega^{AB}(\mathbf{R}^*)_{CD} - \omega^{CD}(\mathbf{R}^*)_{AB}$.

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