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Supplement to the paper "Asymptotic expansions in multi-group analysis of moment structures with an application to linearized estimators" and errata for the paper on the ADF chi-square statistic.

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This note is to supplement Ogasawara (2011), and gives errata for Ogasawara (2009).

1. Supplement to Ogasawara (2011)

Let

$$\mathbf{\Omega}^{-1} = (\omega^{AB}) = \begin{pmatrix} \mathbf{\Omega}^{(k)} & \mathbf{\Omega}^{(k,2k)} \\ \mathbf{\Omega}^{(2k,k)} & \mathbf{\Omega}^{(2k)} \end{pmatrix} = (\mathbf{\Omega}^{(\cdot k)} \;\; \mathbf{\Omega}^{(\cdot 2k)}) = \begin{pmatrix} \mathbf{\Omega}^{(k\cdot)} \\ \mathbf{\Omega}^{(2k\cdot)} \end{pmatrix}$$

$(A, B = 1,\ldots, G^* p^{**})$,

$\mathbf{\Omega}^{-1}_{(2k|k)} = \mathbf{\Omega}^{(2k)}$ and $\mathrm{E}_{\mathbf{u}|\cdot}(\cdot) = \mathrm{E}_{\mathbf{u}_{(k)}|\mathbf{\Gamma}}\mathrm{E}_{\mathbf{u}_{(2k)}|\mathbf{\Omega}_{(2k|k)},\mathbf{u}_{(k)}}(\cdot)$. Let $(\cdot)_{(A-D)} = (\cdot)_{(ABCD)}$ and $(\cdot)_{(A-F)} = (\cdot)_{(ABCDEF)}$. Then, the expectation of $\mathbf{h}_{(2)}$ with the redefinition of $\mathbf{u}_{(2k)} \sim N(\mathbf{\Omega}_{(2k,k)}\mathbf{\Omega}_{\pi}^{-1}\mathbf{u}_{(k)}, \mathbf{\Omega}_{(2k|k)})$ is

$$\mathrm{E}_{\mathbf{u}|\cdot}(\mathbf{h}_{(2)})_{(AB)} = \{\mathbf{\Omega}^{-1}\mathrm{E}_{\mathbf{u}|\cdot}(\mathbf{uu}')\mathbf{\Omega}^{-1}\}_{AB} - \omega^{AB}$$

$$= \left\{\mathbf{\Omega}^{-1}\mathrm{E}_{\mathbf{u}(k)|\mathbf{\Gamma}}\begin{pmatrix} \mathbf{u}_{(k)}\mathbf{u}_{(k)}' & \mathbf{u}_{(k)}\mathbf{u}_{(k)}'\mathbf{\Omega}_{\pi}^{-1}\mathbf{\Omega}_{(k,2k)} \\ \mathbf{\Omega}_{(2k,k)}\mathbf{\Omega}_{\pi}^{-1}\mathbf{u}_{(k)}\mathbf{u}_{(k)}' & \mathbf{\Omega}_{(2k,k)}\mathbf{\Omega}_{\pi}^{-1}\mathbf{u}_{(k)}\mathbf{u}_{(k)}'\mathbf{\Omega}_{\pi}^{-1}\mathbf{\Omega}_{(k,2k)} + \mathbf{\Omega}_{(2k|k)} \end{pmatrix}\mathbf{\Omega}^{-1}\right\}_{AB} - \omega^{AB}$$

$$= \left\{\mathbf{\Omega}^{-1}\begin{pmatrix} \mathbf{\Gamma} & \mathbf{\Gamma}\mathbf{\Omega}_{\pi}^{-1}\mathbf{\Omega}_{(k,2k)} \\ \mathbf{\Omega}_{(2k,k)}\mathbf{\Omega}_{\pi}^{-1}\mathbf{\Gamma} & \mathbf{\Omega}_{(2k,k)}\mathbf{\Omega}_{\pi}^{-1}\mathbf{\Gamma}\mathbf{\Omega}_{\pi}^{-1}\mathbf{\Omega}_{(k,2k)} + \mathbf{\Omega}_{(2k|k)} \end{pmatrix}\mathbf{\Omega}^{-1}\right\}_{AB} - \omega^{AB}.$$

Define $\mathbf{P}$, $\mathbf{P}_1, \mathbf{P}_2, \mathbf{Q}_1, \mathbf{Q}_2$ and $\mathbf{\Phi}$ as follows:

$$\mathbf{\Gamma} = \mathbf{\Omega}_{\pi}^{1/2}(\mathbf{I}_{G^*p^*} - 2\mathrm{i}t\mathbf{\Omega}_{\pi}^{1/2}\mathbf{J}_0\mathbf{\Omega}_{\pi}^{1/2})^{-1}\mathbf{\Omega}_{\pi}^{1/2} = \mathbf{\Omega}_{\pi}^{1/2}(\phi\mathbf{P}_1\mathbf{P}_1' + \mathbf{P}_2\mathbf{P}_2')\mathbf{\Omega}_{\pi}^{1/2}$$
$$= \mathbf{\Omega}_{\pi}^{1/2}\mathbf{P\Phi P}'\mathbf{\Omega}_{\pi}^{1/2} = \mathbf{\Omega}_{\pi}^{1/2}(\phi\mathbf{Q}_1 + \mathbf{Q}_2)\mathbf{\Omega}_{\pi}^{1/2},$$
$$\mathbf{P} = (\mathbf{P}_1\ \mathbf{P}_2),\ \mathbf{PP}' = \mathbf{I}_{G^*p^*}, \mathbf{\Phi} = \begin{pmatrix} \phi\mathbf{I}_{p^+} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_q \end{pmatrix}, \mathbf{Q}_1 = \mathbf{P}_1\mathbf{P}_1', \mathbf{Q}_2 = \mathbf{P}_2\mathbf{P}_2'.$$

Then,

$$\mathrm{E}_{\mathbf{u}|\bullet}(\mathbf{h}_{(2)})_{(AB)} = (\mathbf{\Omega}^{(\bullet k)}\mathbf{\Gamma}\,\mathbf{\Omega}^{(k\bullet)})_{AB} + (\mathbf{\Omega}^{(\bullet 2k)}\mathbf{\Omega}_{(2k,k)}\mathbf{\Omega}_{\pi}^{-1}\mathbf{\Gamma}\,\mathbf{\Omega}^{(k\bullet)})_{AB}$$
$$+(\mathbf{\Omega}^{(\bullet k)}\mathbf{\Gamma}\mathbf{\Omega}_{\pi}^{-1}\,\mathbf{\Omega}_{(k,2k)}\mathbf{\Omega}^{(2k\bullet)})_{AB} + (\mathbf{\Omega}^{(\bullet 2k)}\mathbf{\Omega}_{(2k,k)}\mathbf{\Omega}_{\pi}^{-1}\mathbf{\Gamma}\mathbf{\Omega}_{\pi}^{-1}\,\mathbf{\Omega}_{(k,2k)}\mathbf{\Omega}^{(2k\bullet)})_{AB}$$
$$+(\mathbf{\Omega}^{(\bullet 2k)}\,\mathbf{\Omega}_{(2k|k)}\,\mathbf{\Omega}^{(2k\bullet)})_{AB} - \omega^{AB}$$
$$= \phi\{\ \mathbf{\Omega}^{(\bullet k)}\mathbf{\Omega}_{\pi}^{1/2}\mathbf{Q}_1\mathbf{\Omega}_{\pi}^{1/2}\,\mathbf{\Omega}^{(k\bullet)} + \mathbf{\Omega}^{(\bullet 2k)}\mathbf{\Omega}_{(2k,k)}\mathbf{\Omega}_{\pi}^{-1/2}\mathbf{Q}_1\mathbf{\Omega}_{\pi}^{1/2}\,\mathbf{\Omega}^{(k\bullet)}$$
$$+\mathbf{\Omega}^{(\bullet k)}\mathbf{\Omega}_{\pi}^{1/2}\mathbf{Q}_1\mathbf{\Omega}_{\pi}^{-1/2}\mathbf{\Omega}_{(k,2k)}\mathbf{\Omega}^{(\bullet 2k)}$$
$$+\mathbf{\Omega}^{(\bullet 2k)}\mathbf{\Omega}_{(2k,k)}\mathbf{\Omega}_{\pi}^{-1/2}\mathbf{Q}_1\,\mathbf{\Omega}_{\pi}^{-1/2}\mathbf{\Omega}_{(k,2k)}\mathbf{\Omega}^{(\bullet 2k)}\ \}_{AB}$$
$$+\{\ \mathbf{\Omega}^{(\bullet k)}\mathbf{\Omega}_{\pi}^{1/2}\mathbf{Q}_2\mathbf{\Omega}_{\pi}^{1/2}\,\mathbf{\Omega}^{(k\bullet)} + \mathbf{\Omega}^{(\bullet 2k)}\mathbf{\Omega}_{(2k,k)}\mathbf{\Omega}_{\pi}^{-1/2}\mathbf{Q}_2\mathbf{\Omega}_{\pi}^{1/2}\,\mathbf{\Omega}^{(k\bullet)}$$
$$+\mathbf{\Omega}^{(\bullet k)}\mathbf{\Omega}_{\pi}^{1/2}\mathbf{Q}_2\mathbf{\Omega}_{\pi}^{-1/2}\mathbf{\Omega}_{(k,2k)}\mathbf{\Omega}^{(\bullet 2k)}$$
$$+\mathbf{\Omega}^{(\bullet 2k)}\mathbf{\Omega}_{(2k,k)}\mathbf{\Omega}_{\pi}^{-1/2}\mathbf{Q}_1\,\mathbf{\Omega}_{\pi}^{-1/2}\mathbf{\Omega}_{(k,2k)}\mathbf{\Omega}^{(\bullet 2k)}$$
$$+\mathbf{\Omega}^{(\bullet 2k)}\mathbf{\Omega}_{(2k|k)}\mathbf{\Omega}^{(\bullet 2k)}\ \}_{AB} - \omega^{AB}$$
$$\equiv \phi(\mathbf{Q}_1^*)_{AB} + (\mathbf{Q}_2^*)_{AB} + (\mathbf{\Omega}^*)_{AB} - \omega^{AB}$$
$$\equiv \phi(\mathbf{Q}_1^*)_{AB} + (\mathbf{R}^*)_{AB} - \omega^{AB}.$$

Similarly, we have

$$\mathrm{E}_{\mathbf{u}|\bullet}(\mathbf{h}_{(4)})_{(A-D)} = \phi^2\sum\nolimits^3(\mathbf{Q}_1^*)_{AB}(\mathbf{Q}_1^*)_{CD} + \phi\sum\nolimits^6\{(\mathbf{R}^*)_{AB} - \omega^{AB}\}(\mathbf{Q}_1^*)_{CD}$$
$$+\sum\nolimits^3(\mathbf{R}^*)_{AB}(\mathbf{R}^*)_{CD} - \sum\nolimits^6\omega^{AB}(\mathbf{R}^*)_{CD} + \sum\nolimits^3\omega^{AB}\omega^{CD},$$

$$\mathrm{E}_{\mathbf{u}|\cdot}(\mathbf{h}_{(6)})_{(A-F)} = \phi^3 \sum^{15} (\mathbf{Q}_1^*)_{AB} (\mathbf{Q}_1^*)_{CD} (\mathbf{Q}_1^*)_{EF}$$
$$+\phi^2 \sum^{45} \{(\mathbf{R}^*)_{AB} - \omega^{AB}\} (\mathbf{Q}_1^*)_{CD} (\mathbf{Q}_1^*)_{EF}$$
$$+\phi \sum^{45} \{(\mathbf{R}^*)_{AB} (\mathbf{R}^*)_{CD} - \omega^{AB} (\mathbf{R}^*)_{CD} - \omega^{CD} (\mathbf{R}^*)_{AB} + \omega^{AB} \omega^{CD}\} (\mathbf{Q}_1^*)_{EF}$$
$$+\sum^{15} (\mathbf{R}^*)_{AB} (\mathbf{R}^*)_{CD} (\mathbf{R}^*)_{EF} - \sum^{45} \omega^{AB} (\mathbf{R}^*)_{CD} (\mathbf{R}^*)_{EF}$$
$$+\sum^{45} \omega^{AB} \omega^{CD} (\mathbf{R}^*)_{EF} - \sum^{15} \omega^{AB} \omega^{CD} \omega^{EF}$$
$$(A, B, C, D, E, F = 1, \ldots, G^* p^{**}).$$

Define

$$A_{A-F}^{(1)} = \sum^{15} (\mathbf{Q}_1^*)_{AB} (\mathbf{Q}_1^*)_{CD} (\mathbf{Q}_1^*)_{EF}, \quad A_{A-D}^{(2)} = \sum^{3} (\mathbf{Q}_1^*)_{AB} (\mathbf{Q}_1^*)_{CD},$$
$$A_{A-F}^{(3)} = \sum^{45} \{(\mathbf{R}^*)_{AB} - \omega^{AB}\} (\mathbf{Q}_1^*)_{CD} (\mathbf{Q}_1^*)_{EF},$$
$$A_{A-D}^{(4)} = \sum^{6} \{(\mathbf{R}^*)_{AB} - \omega^{AB}\} (\mathbf{Q}_1^*)_{CD},$$
$$A_{A-F}^{(5)} = \sum^{45} \{(\mathbf{R}^*)_{AB} (\mathbf{R}^*)_{CD} - \omega^{AB} (\mathbf{R}^*)_{CD} - \omega^{CD} (\mathbf{R}^*)_{AB} + \omega^{AB} \omega^{CD}\} (\mathbf{Q}_1^*)_{EF},$$
$$A_{AB}^{(6)} = (\mathbf{R}^*)_{AB} - \omega^{AB},$$
$$A_{A-D}^{(7)} = \sum^{3} (\mathbf{R}^*)_{AB} (\mathbf{R}^*)_{CD} - \sum^{6} \omega^{AB} (\mathbf{R}^*)_{CD} + \sum^{3} \omega^{AB} \omega^{CD},$$
$$A_{A-F}^{(8)} = \sum^{15} (\mathbf{R}^*)_{AB} (\mathbf{R}^*)_{CD} (\mathbf{R}^*)_{EF} - \sum^{45} \omega^{AB} (\mathbf{R}^*)_{CD} (\mathbf{R}^*)_{EF}$$
$$+\sum^{45} \omega^{AB} \omega^{CD} (\mathbf{R}^*)_{EF} - \sum^{15} \omega^{AB} \omega^{CD} \omega^{EF},$$
$$B_{A-D}^{(1)} = \sum^{3} (\mathbf{\Omega Q}_1^*)_{AD} (\mathbf{\Omega Q}_1^* \mathbf{\Omega})_{BC},$$
$$B_{A-D}^{(2)} = \sum^{3} \left\{ (\mathbf{\Omega Q}_1^*)_{AD} (\mathbf{\Omega R}^* \mathbf{\Omega})_{BC} + (\mathbf{\Omega R}^*)_{AD} (\mathbf{\Omega Q}_1^* \mathbf{\Omega})_{BC} \right\},$$
$$B_{A-D}^{(3)} = \sum^{3} (\mathbf{\Omega R}^*)_{AD} (\mathbf{\Omega R}^* \mathbf{\Omega})_{BC},$$
$$B_{A-F}^{(4)} = \sum^{6} (\mathbf{\Omega Q}_1^*)_{AD} (\mathbf{\Omega Q}_1^*)_{BE} (\mathbf{\Omega Q}_1^*)_{CF} + \sum^{9} (\mathbf{\Omega Q}_1^* \mathbf{\Omega})_{AB} (\mathbf{Q}_1^*)_{DE} (\mathbf{\Omega Q}_1^*)_{CF},$$
$$B_{A-F}^{(5)} = \sum^{18} (\mathbf{\Omega Q}_1^*)_{AD} (\mathbf{\Omega Q}_1^*)_{BE} (\mathbf{\Omega R}^*)_{CF} + \sum^{27} (\mathbf{\Omega Q}_1^* \mathbf{\Omega})_{AB} (\mathbf{Q}_1^*)_{DE} (\mathbf{\Omega R}^*)_{CF}$$
$$-\sum^{9} (\mathbf{\Omega Q}_1^*)_{AD} (\mathbf{\Omega Q}_1^* \mathbf{\Omega})_{BC} \, \omega^{EF},$$

$$B^{(6)}_{A-F} = \sum^{18} (\mathbf{\Omega Q}^*_1)_{AD} (\mathbf{\Omega R}^*)_{BE} (\mathbf{\Omega R}^*)_{CF} + \sum^{27} (\mathbf{\Omega Q}^*_1\mathbf{\Omega})_{AB} (\mathbf{R}^*)_{DE} (\mathbf{\Omega R}^*)_{CF}$$
$$- \sum^{18} (\mathbf{\Omega Q}^*_1)_{AD} (\mathbf{\Omega R}^*\mathbf{\Omega})_{BC}\, \omega^{EF},$$
$$B^{(7)}_{A-F} = \sum^{6} (\mathbf{\Omega R}^*)_{AD} (\mathbf{\Omega R}^*)_{BE} (\mathbf{\Omega R}^*)_{CF} + \sum^{9} (\mathbf{\Omega R}^*\mathbf{\Omega})_{AB} (\mathbf{R}^*)_{DE} (\mathbf{\Omega R}^*)_{CF}$$
$$- \sum^{9} (\mathbf{\Omega R}^*)_{AD} (\mathbf{\Omega R}^*\mathbf{\Omega})_{BC}\, \omega^{EF},$$
$$B^{(8)}_{A-D} = \sum^{3} (\mathbf{\Omega Q}^*_1\mathbf{\Omega})_{AB} (\mathbf{\Omega Q}^*_1\mathbf{\Omega})_{CD}, B^{(9)}_{A-D} = \sum^{6} (\mathbf{\Omega Q}^*_1\mathbf{\Omega})_{AB} (\mathbf{\Omega R}^*\mathbf{\Omega})_{CD},$$
$$B^{(10)}_{A-D} = \sum^{3} (\mathbf{\Omega R}^*\mathbf{\Omega})_{AB} (\mathbf{\Omega R}^*\mathbf{\Omega})_{CD},$$
$$C^{(1)}_{A-F} = \sum^{15} (\mathbf{\Omega Q}^*_1\mathbf{\Omega})_{AB} (\mathbf{\Omega Q}^*_1\mathbf{\Omega})_{CD} (\mathbf{\Omega Q}^*_1\mathbf{\Omega})_{EF},$$
$$C^{(2)}_{A-F} = \sum^{45} (\mathbf{\Omega Q}^*_1\mathbf{\Omega})_{AB} (\mathbf{\Omega Q}^*_1\mathbf{\Omega})_{CD} (\mathbf{\Omega R}^*\mathbf{\Omega})_{EF},$$
$$C^{(3)}_{A-F} = \sum^{45} (\mathbf{\Omega Q}^*_1\mathbf{\Omega})_{AB} (\mathbf{\Omega R}^*\mathbf{\Omega})_{CD} (\mathbf{\Omega R}^*\mathbf{\Omega})_{EF},$$
$$C^{(4)}_{A-F} = \sum^{15} (\mathbf{\Omega R}^*\mathbf{\Omega})_{AB} (\mathbf{\Omega R}^*\mathbf{\Omega})_{CD} (\mathbf{\Omega R}^*\mathbf{\Omega})_{EF}$$
$$(A,\ldots,F = 1,\ldots,G * p^{**}).$$

Then, after some algebra (see Ogasawara, 2009),

$$C_T(t^*) = \phi^{p^+/2}\{1 + N_\bullet^{-1}(a^{(3)}\phi^3 + a^{(2)}\phi^2 + a^{(1)}\phi + a^{(0)}$$
$$+ \mathrm{i}t^* b^{(1)} + (\mathrm{i}t^*)^2 b^{(2)})\} + O(N_\bullet^{-2}),$$

where

$$a^{(3)} = \sum_{A-F} \left[\frac{1}{72} \left\{ (\boldsymbol{\beta}^{<2>}_{(3)})_{(A-F)} A^{(1)}_{A-F} + \frac{\partial^3 F_0}{\partial \tau_A \partial \tau_B \partial \tau_C} (\boldsymbol{\beta}_{(3)})_{(DEF)} B^{(4)}_{A-F} \right\} \right.$$
$$\left. + \frac{1}{288} \frac{\partial^3 F_0}{\partial \tau_A \partial \tau_B \partial \tau_C} \frac{\partial^3 F_0}{\partial \tau_D \partial \tau_E \partial \tau_F} C^{(1)}_{A-F} \right],$$

$$a^{(2)}=\sum_{A-D}\left\{\left(\frac{1}{24}\boldsymbol{\beta}_{(4)}+\frac{1}{6}\boldsymbol{\beta}_{(1)}\otimes\boldsymbol{\beta}_{(3)}\right)_{(A-D)}A^{(2)}_{A-D}\right.$$

$$\left.+\frac{1}{12}\frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}(\boldsymbol{\beta}_{(1)})_D B^{(1)}_{A-D}+\frac{1}{48}\frac{\partial^4 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C\partial\tau_D}B^{(8)}_{A-D}\right\}$$

$$+\sum_{A-F}\left[\frac{1}{72}\left\{(\boldsymbol{\beta}^{<2>}_{(3)})_{(A-F)}A^{(3)}_{A-F}+\frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}(\boldsymbol{\beta}_{(3)})_{(DEF)}(-B^{(4)}_{A-F}+B^{(5)}_{A-F})\right\}\right.$$

$$\left.+\frac{1}{288}\frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}\frac{\partial^3 F_0}{\partial\tau_D\partial\tau_E\partial\tau_F}(-2C^{(1)}_{A-F}+C^{(2)}_{A-F})\right],$$

$$a^{(1)}=\frac{1}{2}\sum_{AB}(\boldsymbol{\beta}_{(\Delta 2)}+\boldsymbol{\beta}^{<2>}_{(1)})_{(AB)}(\mathbf{Q}^*_1)_{AB}+\sum_{A-D}\left\{\left(\frac{1}{24}\boldsymbol{\beta}_{(4)}+\frac{1}{6}\boldsymbol{\beta}_{(1)}\otimes\boldsymbol{\beta}_{(3)}\right)_{(A-D)}A^{(4)}_{A-D}\right.$$

$$\left.+\frac{1}{12}\frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}(\boldsymbol{\beta}_{(1)})_D(-B^{(1)}_{A-D}+B^{(2)}_{A-D})+\frac{1}{48}\frac{\partial^4 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C\partial\tau_D}(-B^{(8)}_{A-D}+B^{(9)}_{A-D})\right\}$$

$$+\sum_{A-F}\left[\frac{1}{72}\left\{(\boldsymbol{\beta}^{<2>}_{(3)})_{(A-F)}A^{(5)}_{A-F}+\frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}(\boldsymbol{\beta}_{(3)})_{(DEF)}(-B^{(5)}_{A-F}+B^{(6)}_{A-F})\right\}\right.$$

$$\left.+\frac{1}{288}\frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}\frac{\partial^3 F_0}{\partial\tau_D\partial\tau_E\partial\tau_F}(C^{(1)}_{A-F}-2C^{(2)}_{A-F}+C^{(3)}_{A-F})\right],$$

$$a^{(0)}=\frac{1}{2}\sum_{AB}(\boldsymbol{\beta}_{(\Delta 2)}+\boldsymbol{\beta}^{<2>}_{(1)})_{(AB)}A^{(6)}_{AB}+\sum_{A-D}\left\{\left(\frac{1}{24}\boldsymbol{\beta}_{(4)}+\frac{1}{6}\boldsymbol{\beta}_{(1)}\otimes\boldsymbol{\beta}_{(3)}\right)_{(A-D)}A^{(7)}_{A-D}\right.$$

$$\left.-\frac{1}{12}\frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}(\boldsymbol{\beta}_{(1)})_D B^{(2)}_{A-D}-\frac{1}{48}\frac{\partial^4 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C\partial\tau_D}B^{(9)}_{A-D}\right\}$$

$$+\sum_{A-F}\left[\frac{1}{72}\left\{(\boldsymbol{\beta}^{<2>}_{(3)})_{(A-F)}A^{(8)}_{A-F}-\frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}(\boldsymbol{\beta}_{(3)})_{(DEF)}B^{(6)}_{A-F}\right\}\right.$$

$$\left.+\frac{1}{288}\frac{\partial^3 F_0}{\partial\tau_A\partial\tau_B\partial\tau_C}\frac{\partial^3 F_0}{\partial\tau_D\partial\tau_E\partial\tau_F}(C^{(2)}_{A-F}-C^{(3)}_{A-F})\right],$$

$$b^{(1)} = \sum_{A-D} \left\{ \frac{1}{6} \frac{\partial^3 F_0}{\partial \tau_A \partial \tau_B \partial \tau_C} (\boldsymbol{\beta}_{(1)})_D B^{(3)}_{A-D} + \frac{1}{24} \frac{\partial^4 F_0}{\partial \tau_A \partial \tau_B \partial \tau_C \partial \tau_D} B^{(10)}_{A-D} \right\}$$

$$+ \sum_{A-F} \left\{ \frac{1}{36} \frac{\partial^3 F_0}{\partial \tau_A \partial \tau_B \partial \tau_C} (\boldsymbol{\beta}_{(3)})_{(DEF)} B^{(7)}_{A-F} - \frac{1}{144} \frac{\partial^3 F_0}{\partial \tau_A \partial \tau_B \partial \tau_C} \frac{\partial^3 F_0}{\partial \tau_D \partial \tau_E \partial \tau_F} C^{(3)}_{A-F} \right\},$$

$$b^{(2)} = \frac{1}{72} \sum_{A-F} \frac{\partial^3 F_0}{\partial \tau_A \partial \tau_B \partial \tau_C} \frac{\partial^3 F_0}{\partial \tau_D \partial \tau_E \partial \tau_F} C^{(4)}_{A-F}.$$

2. Errata for Ogasawara (2009)

Ogasawara's (2009, Equations (4.4) and (4.5)) result with k=2 and without mean structures corresponding to the result in the previous section is a special case of this note, and should be corrected by using the definitions of $\mathbf{Q}_1^*$, $\mathbf{Q}_2^*$, $\boldsymbol{\Omega}^*$ and $\mathbf{R}^*$ given earlier. See also Ogasawara (2010) for an additional minor erratum in Ogasawara (2009, Equations (4.6) and (4.7)).

References

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