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Supplement to the papers on "the polyserial correlation coefficients" and "discrepancy functions for general covariance structures"

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Parts A and B of this note are to supplement Ogasawara (2011, 2010), respectively.

Part A

0. Derivation of the inverse expansion

Let $\mathbf{L} = \partial^2 \overline{l} / \partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0 '$. Then, from (3.1) the inverse expansion of $\hat{\boldsymbol{\theta}}$ is

$$\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 = -\mathbf{L}^{-1} \frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0} - \frac{1}{2} \mathbf{L}^{-1} \frac{\partial^3 \overline{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0 ')^{<2>}} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)^{<2>} - \frac{1}{6} \mathbf{L}^{-1} \frac{\partial^4 \overline{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0 ')^{<3>}} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)^{<3>} + O_p(N^{-2})$$

$$= -\mathbf{L}^{-1} \frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0} - \frac{1}{2} \mathbf{L}^{-1} \frac{\partial^3 \overline{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0 ')^{<2>}} \left(\mathbf{L}^{-1} \frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>}$$

$$- \frac{1}{2} \mathbf{L}^{-1} \frac{\partial^3 \overline{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0 ')^{<2>}} \left[\left(\mathbf{L}^{-1} \frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0} \right) \otimes \left\{ \mathbf{L}^{-1} \frac{\partial^3 \overline{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0 ')^{<2>}} \left(\mathbf{L}^{-1} \frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>} \right\} \right]$$

$$+ \frac{1}{6} \frac{\partial^4 \overline{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0 ')^{<3>}} \left(\mathbf{L}^{-1} \frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0} \right)^{<3>} + O_p(N^{-2}).$$

Let $\mathrm{E}(\mathbf{L}) = \boldsymbol{\Lambda} = -\mathbf{I}$, where $\mathbf{I}$ is the information matrix per observation, and let $\mathbf{L} = \boldsymbol{\Lambda} + \mathbf{M}$, where $\mathbf{M} = O_p(N^{-1/2})$. Then, we obtain

$$\mathbf{L}^{-1} = \boldsymbol{\Lambda}^{-1} - \boldsymbol{\Lambda}^{-1} \mathbf{M} \boldsymbol{\Lambda}^{-1} + \boldsymbol{\Lambda}^{-1} \mathbf{M} \boldsymbol{\Lambda}^{-1} \mathbf{M} \boldsymbol{\Lambda}^{-1} + O_p(N^{-3/2}).$$

The above results give

$$\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 = \left[-\boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right]_{O_p(N^{-1/2})}$$

$$+ \left[\boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} - \frac{1}{2}\boldsymbol{\Lambda}^{-1}\mathrm{E}\left(\frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0 ')^{<2>}} \right) \left(\boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>} \right]_{O_p(N^{-1})}$$

$$+ \left[-\boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} + \frac{1}{2}\boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1}\mathrm{E}\left(\frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0 ')^{<2>}} \right) \left(\boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>} \right.$$

$$+ \boldsymbol{\Lambda}^{-1}\mathrm{E}\left(\frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0 ')^{<2>}} \right) \left\{ \left(\boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right) \otimes \left(\boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right) \right\}$$

$$- \frac{1}{2}\boldsymbol{\Lambda}^{-1} \left\{ \frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0 ')^{<2>}} - \mathrm{E}\left(\frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0 ')^{<2>}} \right) \right\} \left(\boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>}$$

$$- \frac{1}{2}\boldsymbol{\Lambda}^{-1}\mathrm{E}\left(\frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0 ')^{<2>}} \right) \left[\left(\boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right) \otimes \left\{ \boldsymbol{\Lambda}^{-1}\mathrm{E}\left(\frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0 ')^{<2>}} \right) \left(\boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>} \right\} \right]$$

$$\left. + \frac{1}{6}\boldsymbol{\Lambda}^{-1}\mathrm{E}\left(\frac{\partial^4 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0 ')^{<3>}} \right) \left(\boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<3>} \right]_{O_p(N^{-3/2})} + O_p(N^{-2})$$

$$\equiv \sum_{i=1}^{3} \boldsymbol{\Lambda}^{(i)} \mathbf{l}_0^{(i)} + O_p(N^{-2}),$$

where $[\cdot]_{O_p(\cdot)}$ indicates that the sum of the terms in brackets is of order $O_p(\cdot)$ for clarity.

1. Expectations of the log likelihood derivatives

Let $\boldsymbol{\Lambda} = \mathrm{Bdiag}(\boldsymbol{\Lambda}_x, \boldsymbol{\Lambda}_z)$ and $\mathbf{I} = \mathrm{Bdiag}(\mathbf{I}_x, \mathbf{I}_z)$, where $\boldsymbol{\Lambda} = -\mathbf{I}$, $\mathrm{Bdiag}(\cdot)$ denotes the block diagonal matrix with the diagonal blocks being the matrices in parentheses, and $\mathbf{I}$ is the information matrix per observation.

1.1 Information matrix

(1) $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$

$$\frac{\partial l_x^*}{\partial \boldsymbol{\mu}} = \sum_{i=1}^{N} \boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu}),$$

$$\frac{\partial l_x^*}{\partial \sigma_{ab}} = \frac{2-\delta_{ab}}{2} \sum_{i=1}^{N} [-\sigma^{ab} + \{\boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu})\}_a \{\boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu})\}_b],$$

$$\frac{\partial^2 l_x^*}{\partial \boldsymbol{\mu} \partial \boldsymbol{\mu}'} = -N \boldsymbol{\Sigma}^{-1},$$

$$\frac{\partial^2 l_x^*}{\partial \sigma_{ab} \partial \boldsymbol{\mu}} = -\frac{2-\delta_{ab}}{2} \sum_{i=1}^{N} [(\boldsymbol{\Sigma}^{-1})_{\cdot a} \{\boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu})\}_b + (\boldsymbol{\Sigma}^{-1})_{\cdot b} \{\boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu})\}_a],$$

$$\frac{\partial^2 l_x^*}{\partial \sigma_{ab} \partial \sigma_{cd}} = \frac{1}{4}(2-\delta_{ab})(2-\delta_{cd}) \sum_{i=1}^{N} \Bigg[\sigma^{ac}\sigma^{db} + \sigma^{ad}\sigma^{cb} - \sum^{4} \sigma^{ac} \{\boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu})\}_d \{\boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu})\}_b \Bigg],$$

where $\overset{k}{\boldsymbol{\Sigma}}$ denotes the sum of *k* terms with similar patterns. Then, we obtain

$$-N(\mathbf{I}_x)_{\boldsymbol{\mu}\boldsymbol{\mu}'} = N(\boldsymbol{\Lambda}_x)_{\boldsymbol{\mu}\boldsymbol{\mu}'} = \mathrm{E}\left(\frac{\partial^2 l_x^*}{\partial \boldsymbol{\mu} \partial \boldsymbol{\mu}'} \right) = \frac{\partial^2 l_x^*}{\partial \boldsymbol{\mu} \partial \boldsymbol{\mu}'} = -N \boldsymbol{\Sigma}^{-1},$$

$$-N(\mathbf{I}_x)_{\sigma_{ab}\boldsymbol{\mu}'} = 0,$$

$$-N(\mathbf{I}_x)_{\sigma_{ab}\sigma_{cd}} = N(\boldsymbol{\Lambda}_x)_{\sigma_{ab}\sigma_{cd}} = \mathrm{E}\left(\frac{\partial^2 l_x^*}{\partial \sigma_{ab} \partial \sigma_{cd}} \right) = -\frac{N}{4}(2-\delta_{ab})(2-\delta_{cd}) \times (\sigma^{ac}\sigma^{db} + \sigma^{ad}\sigma^{cb}),$$

that is,

$$(\boldsymbol{\Lambda}_x)_{\boldsymbol{\mu}\boldsymbol{\mu}'} = -\boldsymbol{\Sigma}^{-1}, \quad (\boldsymbol{\Lambda}_x)_{\boldsymbol{\mu}\sigma_{ab}} = 0,$$

$$(\boldsymbol{\Lambda}_x)_{\sigma_{ab}\sigma_{cd}} = -\frac{1}{4}(2-\delta_{ab})(2-\delta_{cd})(\sigma^{ac}\sigma^{db} + \sigma^{ad}\sigma^{cb}),$$

$$(r \ge a \ge b \ge 1;\ r \ge c \ge d \ge 1)$$

where, e.g., $(\cdot)_{\mathbf{u}\mathbf{v}'}$ denotes the submatrix of the matrix in parentheses corresponding to the product $\mathbf{u}\mathbf{v}'$, and the ranges of (*a*, *b*), (*c*, *d*), and similar pairs will be used hereafter in similar situations.

(2) $\boldsymbol{\beta}(=(\beta_1,\ldots,\beta_{K-1})')$ **and** $\boldsymbol{\xi}(=(\xi_1,\ldots,\xi_r)')$

$$\frac{\partial l_z^*}{\partial \beta_k}=\sum_{i=1}^{N}\frac{1}{\pi_{z_i}}(\delta_{kz_i}\phi_{z_i}-\delta_{k(z_i-1)}\phi_{z_i-1})$$, where

$$\pi_{z_i}=\Phi(\gamma_{z_i})-\Phi(\gamma_{z_i-1}),\quad \Phi(\gamma_{z_i})=\int_{-\infty}^{\gamma_{z_i}}\phi(\gamma)d\gamma,$$

$$\phi(\gamma_{z_i})=\phi_{z_i}=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\gamma_{z_i}^2}{2}\right\},$$

$$\gamma_{z_i}=\beta_{z_i}+\boldsymbol{\xi}'\mathbf{x}_i,\quad (\phi_0=\phi_K=\gamma_0\phi_0=\gamma_K\phi_K=0),$$

$$\frac{\partial l_z^*}{\partial \boldsymbol{\xi}}=\sum_{i=1}^{N}\frac{\mathbf{x}_i}{\pi_{z_i}}(\phi_{z_i}-\phi_{z_i-1}),$$

$$\frac{\partial^2 l_z^*}{\partial \beta_k \partial \beta_l}=\sum_{i=1}^{N}\left\{\frac{1}{\pi_{z_i}}(-\delta_{klz_i}\gamma_{z_i}\phi_{z_i}+\delta_{kl(z_i-1)}\gamma_{z_i-1}\phi_{z_i-1})\right.$$
$$\left.-\frac{1}{\pi_{z_i}^2}(\delta_{kz_i}\phi_{z_i}-\delta_{k(z_i-1)}\phi_{z_i-1})(\delta_{lz_i}\phi_{z_i}-\delta_{l(z_i-1)}\phi_{z_i-1})\right\},$$

where $\delta_{klz_i}=\delta_{kl}\delta_{lz_i}$,

$$\frac{\partial^2 l_z^*}{\partial \boldsymbol{\xi}\partial \beta_k}=\sum_{i=1}^{N}\mathbf{x}_i\left\{\frac{1}{\pi_{z_i}}(-\delta_{kz_i}\gamma_{z_i}\phi_{z_i}+\delta_{k(z_i-1)}\gamma_{z_i-1}\phi_{z_i-1})\right.$$
$$\left.-\frac{1}{\pi_{z_i}^2}(\delta_{kz_i}\phi_{z_i}-\delta_{k(z_i-1)}\phi_{z_i-1})(\phi_{z_i}-\phi_{z_i-1})\right\},$$

$$\frac{\partial^2 l_z^*}{\partial \boldsymbol{\xi}\partial \boldsymbol{\xi}'}=\sum_{i=1}^{N}\mathbf{x}_i\mathbf{x}_i'\left\{\frac{1}{\pi_{z_i}}(-\gamma_{z_i}\phi_{z_i}+\gamma_{z_i-1}\phi_{z_i-1})-\frac{1}{\pi_{z_i}^2}(\phi_{z_i}-\phi_{z_i-1})^2\right\},$$

$$-N(\mathbf{I}_z)_{\beta_k\beta_l} = N(\mathbf{\Lambda}_z)_{\beta_k\beta_l} = \mathrm{E}_X \mathrm{E}_{Z|X}\left(\frac{\partial^2 l_z^*}{\partial\beta_k\partial\beta_l}\right)$$

$$= N\,\mathrm{E}_X\left(-\delta_{kl}\gamma_k\phi_k + \delta_{kl}\gamma_k\phi_k - \frac{1}{\pi_k}\delta_{kl}\phi_k^2 + \frac{1}{\pi_l}\delta_{(k+1)l}\phi_k\phi_l\right.$$

$$\left. + \frac{1}{\pi_k}\delta_{(l+1)k}\phi_k\phi_l - \frac{1}{\pi_{k+1}}\delta_{kl}\phi_k^2\right)$$

$$= N\,\mathrm{E}_X\left\{-\delta_{kl}\phi_k^2\left(\frac{1}{\pi_k}+\frac{1}{\pi_{k+1}}\right)+\phi_k\phi_l\left(\frac{\delta_{(k+1)l}}{\pi_l}+\frac{\delta_{(l+1)k}}{\pi_k}\right)\right\},$$

where

$$\mathrm{E}_X\{\cdot\} = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\{\cdot\}p(\mathbf{x})d\mathbf{x},$$

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{r/2}|\mathbf{\Sigma}|^{1/2}}\exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\},$$

$$-N(\mathbf{I}_z)_{\xi\beta_k} = N(\mathbf{\Lambda}_z)_{\xi\beta_k} = \mathrm{E}_X \mathrm{E}_{Z|X}\left(\frac{\partial^2 l_z^*}{\partial\boldsymbol{\xi}\,\partial\beta_k}\right)$$

$$= N\,\mathrm{E}_X\left\{\mathbf{x}\left(-\frac{\phi_k^2}{\pi_k}+\frac{1}{\pi_k}\phi_k\phi_{k-1}+\frac{1}{\pi_{k+1}}\phi_{k+1}\phi_k-\frac{\phi_k^2}{\pi_{k+1}}\right)\right\},$$

$$-N(\mathbf{I}_z)_{\xi\xi'} = N(\mathbf{\Lambda}_z)_{\xi\xi'} = \mathrm{E}_X \mathrm{E}_{Z|X}\left(\frac{\partial^2 l_z^*}{\partial\boldsymbol{\xi}\,\partial\boldsymbol{\xi}'}\right) = N\,\mathrm{E}_X\left\{\mathbf{x}\,\mathbf{x}'\sum_{a=1}^{K}\frac{-1}{\pi_a}(\phi_a-\phi_{a-1})^2\right\},$$

that is,

$$(\mathbf{\Lambda}_z)_{\beta_k\beta_l} = \mathrm{E}_X\left\{-\delta_{kl}\phi_k^2\left(\frac{1}{\pi_k}+\frac{1}{\pi_{k+1}}\right)+\phi_k\phi_l\left(\frac{\delta_{(k+1)l}}{\pi_l}+\frac{\delta_{(l+1)k}}{\pi_k}\right)\right\},$$

$$(\mathbf{\Lambda}_z)_{\xi\beta_k} = \mathrm{E}_X\left[\mathbf{x}\left\{-\frac{1}{\pi_k}\phi_k(\phi_k-\phi_{k-1})+\frac{1}{\pi_{k+1}}\phi_k(\phi_{k+1}-\phi_k)\right\}\right],$$

$$(\mathbf{\Lambda}_z)_{\xi\xi'} = \mathrm{E}_X\left\{\mathbf{x}\,\mathbf{x}'\sum_{a=1}^{K}\frac{-1}{\pi_a}(\phi_a-\phi_{a-1})^2\right\}.$$

1.2 Expectations of the products of the log likelihood derivatives

Recall $\mathbf{L} = \mathbf{\Lambda} + \mathbf{M}$ and $\bar{l} = N^{-1}l^*$. Let $\mathbf{M} = \mathrm{Bdiag}(\mathbf{M}_x, \mathbf{M}_z)$.

(1) $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$

(1-1) $\mathrm{E}(N\mathbf{l}_0^{(2)}) = N\,\mathrm{E}\left\{\mathrm{v}(\mathbf{M})'\otimes\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0'}, \left(\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0'}\right)^{<2>}\right\}'$ (for α_1)

The nonzero results of

$$N\,\mathrm{E}\left(\frac{\partial^2\bar{l}}{\partial(\boldsymbol{\theta}_x)_A\partial(\boldsymbol{\theta}_x)_B}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_x)_C}\right)=\mathrm{E}_X\left(\frac{\partial^2\ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A\partial(\boldsymbol{\theta}_x)_B}\frac{\partial\ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C}\right),$$ where

$(\cdot)_A$ denotes the A-th element of the vector in parentheses, are

$$\mathrm{E}_X\left(\frac{\partial^2\ln p(\mathbf{x})}{\partial\sigma_{ab}\partial\boldsymbol{\mu}}\frac{\partial\ln p(\mathbf{x})}{\partial\boldsymbol{\mu}'}\right)=-\frac{2-\delta_{ab}}{2}\{(\boldsymbol{\Sigma}^{-1})_{\cdot a}(\boldsymbol{\Sigma}^{-1})_{b\cdot}+(\boldsymbol{\Sigma}^{-1})_{\cdot b}(\boldsymbol{\Sigma}^{-1})_{a\cdot}\},$$

$$\mathrm{E}_X\left(\frac{\partial^2\ln p(\mathbf{x})}{\partial\sigma_{ab}\partial\sigma_{cd}}\frac{\partial\ln p(\mathbf{x})}{\partial\sigma_{ef}}\right)=\frac{1}{8}(2-\delta_{ab})(2-\delta_{cd})(2-\delta_{ef})\{-\sum^{8}\sigma^{ac}\sigma^{de}\sigma^{fb}\}.$$

(1-2) $\mathrm{E}(N^2\mathbf{l}_0^{(1)<3>})$ (for α_3)

The nonzero results of $\mathrm{E}_X\left\{\left(\frac{\partial p(\mathbf{x})}{\partial\boldsymbol{\theta}_x}\right)^{<3>}\right\}$ are

$$\mathrm{E}_X\left(\frac{\partial\ln p(\mathbf{x})}{\partial\boldsymbol{\mu}}\frac{\partial\ln p(\mathbf{x})}{\partial\boldsymbol{\mu}'}\frac{\partial\ln p(\mathbf{x})}{\partial\sigma_{ab}}\right)=\frac{2-\delta_{ab}}{2}\{(\boldsymbol{\Sigma}^{-1})_{\cdot a}(\boldsymbol{\Sigma}^{-1})_{b\cdot}$$
$$+(\boldsymbol{\Sigma}^{-1})_{\cdot b}(\boldsymbol{\Sigma}^{-1})_{a\cdot}\},$$

$$\mathrm{E}_X\left(\frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{ab}}\frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{cd}}\frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{ef}}\right)=\frac{1}{8}(2-\delta_{ab})(2-\delta_{cd})(2-\delta_{ef})$$

$$\times(2\sigma^{ab}\sigma^{cd}\sigma^{ef}-\sigma^{ab}\sum^{3}\sigma^{cd}\sigma^{ef}-\sigma^{cd}\sum^{3}\sigma^{ab}\sigma^{ef}-\sigma^{ef}\sum^{3}\sigma^{ab}\sigma^{cd}$$

$$+\sum^{15}\sigma^{ab}\sigma^{cd}\sigma^{ef})$$

$$=\frac{1}{8}(2-\delta_{ab})(2-\delta_{cd})(2-\delta_{ef})\sum^{8}\sigma^{ac}\sigma^{de}\sigma^{fb}.$$

(1-3) $\mathrm{E}(N^2\mathbf{l}_0^{(1)<2>}\otimes\mathbf{l}_0^{(2)})$ (for α_3)

$$N^2\mathrm{E}\left(\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_x)_A}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_x)_B}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_x)_C}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_x)_D}\right)$$

$$=\sum^{3}\mathrm{E}_X\left(\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_B}\right)\mathrm{E}_X\left(\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D}\right)+O(N^{-1}),$$

$$N^2\mathrm{E}\left[\left\{\frac{\partial^2\bar{l}}{\partial(\boldsymbol{\theta}_x)_A\partial(\boldsymbol{\theta}_x)_B}-\mathrm{E}\left(\frac{\partial^2\bar{l}}{\partial(\boldsymbol{\theta}_x)_A\partial(\boldsymbol{\theta}_x)_B}\right)\right\}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_x)_C}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_x)_D}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_x)_E}\right]$$

$$=\sum^{3}\mathrm{E}_X\left(\frac{\partial^2 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A\partial(\boldsymbol{\theta}_x)_B}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C}\right)\mathrm{E}_X\left(\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_E}\right)+O(N^{-1}).$$

(1-4) $\mathrm{E}(N^2\mathbf{l}_0^{(2)}\mathbf{l}_0^{(2)}{}')$ (for $\alpha_{\Delta 2}$)

Define $\mathrm{E}\{X-\mathrm{E}(\cdot)\}$ as $\mathrm{E}\{X-\mathrm{E}(X)\}$. In

$$N^2\mathrm{E}\left[\left\{\frac{\partial^2\bar{l}}{\partial(\boldsymbol{\theta}_x)_A\partial(\boldsymbol{\theta}_x)_B}-\mathrm{E}(\cdot)\right\}\left\{\frac{\partial^2\bar{l}}{\partial(\boldsymbol{\theta}_x)_C\partial(\boldsymbol{\theta}_x)_D}-\mathrm{E}(\cdot)\right\}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_x)_E}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_x)_F}\right],$$

we derive the nonzero results of

$$\mathrm{E}_X\left[\left\{\frac{\partial^2 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B}-\mathrm{E}_X(\cdot)\right\}\left\{\frac{\partial^2 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C \partial(\boldsymbol{\theta}_x)_D}-\mathrm{E}_X(\cdot)\right\}\right]$$
$$=\mathrm{E}_X[\{(\mathbf{L}_x)_{AB}-(\boldsymbol{\Lambda}_x)_{AB}\}\{(\mathbf{L}_x)_{CD}-(\boldsymbol{\Lambda}_x)_{CD}\}]$$
$$=\mathrm{E}_X[\{(\mathbf{L}_x)_{AB}-(\boldsymbol{\Lambda}_x)_{AB}\}(\mathbf{L}_x)_{CD}],$$

where $\mathbf{L}_x \equiv \frac{\partial^2 \ln p(\mathbf{x})}{\partial \boldsymbol{\theta}_x \partial \boldsymbol{\theta}_x{}'}$ $(\mathbf{L}=\mathrm{Bdiag}(\mathbf{L}_x,\mathbf{L}_z)),$

$$\mathrm{E}_X\left[\left\{\frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{ab} \partial \mu_c}-\mathrm{E}_X(\cdot)\right\}\left\{\frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{de} \partial \mu_f}-\mathrm{E}_X(\cdot)\right\}\right]$$
$$=\mathrm{E}_X\left(\frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{ab} \partial \mu_c}\frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{de} \partial \mu_f}\right)=\frac{1}{4}(2-\delta_{ab})(2-\delta_{de})\sum^{4}\sigma^{ac}\sigma^{bd}\sigma^{ef},$$

$$\mathrm{E}_X\left[\left\{\frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{ab} \partial \sigma_{cd}}-\mathrm{E}_X(\cdot)\right\}\left\{\frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{ef} \partial \sigma_{gh}}-\mathrm{E}_X(\cdot)\right\}\right]$$
$$=\frac{1}{16}(2-\delta_{ab})(2-\delta_{cd})(2-\delta_{ef})(2-\delta_{gh})$$
$$\times\mathrm{E}_X\left[\left\{2\sigma^{ac}\sigma^{db}+2\sigma^{ad}\sigma^{cb}-\sum^{4}\sigma^{ac}\{\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}_d\{\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}_b\right\}\right.$$
$$\left.\times\left\{2\sigma^{eg}\sigma^{hf}+2\sigma^{eh}\sigma^{gf}-\sum^{4}\sigma^{eg}\{\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}_h\{\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}_f\right\}\right]$$
$$=\frac{1}{16}(2-\delta_{ab})(2-\delta_{cd})(2-\delta_{ef})(2-\delta_{gh})\sum^{16}\sigma^{ac}\sigma^{eg}(\sigma^{dh}\sigma^{bf}+\sigma^{df}\sigma^{bh}).$$

(1-5) $\mathrm{E}(N^2\mathbf{l}_0^{(1)}\mathbf{l}_0^{(2)}{}')$ (for $\alpha_{\Delta 2}$)

For $\mathrm{E}(N^2\mathbf{l}_0^{(1)<3>})$, see (1-2). The nonzero results of

$$\mathrm{E}_X\left[\{(\mathbf{L}_x)_{AB}-(\boldsymbol{\Lambda}_x)_{AB}\}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D}\right]$$

$$=\mathrm{E}_X\left\{(\mathbf{L}_x)_{AB}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D}\right\}+(\boldsymbol{\Lambda}_x)_{AB}(\boldsymbol{\Lambda}_x)_{CD}$$

are

$$\mathrm{E}_X\left\{\frac{\partial^2 \ln p(\mathbf{x})}{\partial\sigma_{ab}\partial\mu_c}\frac{\partial \ln p(\mathbf{x})}{\partial\sigma_{de}}\frac{\partial \ln p(\mathbf{x})}{\partial\mu_f}\right\}=-\frac{1}{4}(2-\delta_{ab})(2-\delta_{de})\sum^{4}\sigma^{ca}\sigma^{bd}\sigma^{ef},$$

$$\mathrm{E}_X\left[\left\{\frac{\partial^2 \ln p(\mathbf{x})}{\partial\sigma_{ab}\partial\sigma_{cd}}-\mathrm{E}_X(\cdot)\right\}\frac{\partial \ln p(\mathbf{x})}{\partial\mu_e}\frac{\partial \ln p(\mathbf{x})}{\partial\mu_f}\right]$$

$$=-\frac{1}{4}(2-\delta_{ab})(2-\delta_{cd})\sum^{4}\sigma^{ac}(\sigma^{de}\sigma^{bf}+\sigma^{df}\sigma^{be}),$$

$$\mathrm{E}_X\left[\left\{\frac{\partial^2 \ln p(\mathbf{x})}{\partial\sigma_{ab}\partial\sigma_{cd}}-\mathrm{E}_X(\cdot)\right\}\frac{\partial \ln p(\mathbf{x})}{\partial\sigma_{ef}}\frac{\partial \ln p(\mathbf{x})}{\partial\sigma_{gh}}\right]$$

$$=-\frac{1}{16}(2-\delta_{ab})(2-\delta_{cd})(2-\delta_{ef})(2-\delta_{gh})\sum^{4}\sum^{8}\sigma^{ac}\sigma^{de}\sigma^{fg}\sigma^{hb}.$$

(1-6) $\mathrm{E}(N^2\mathbf{l}_0^{(1)}\mathbf{l}_0^{(3)}{}')$ (for $\alpha_{\Delta 2}$)

For $\mathrm{E}(N^2\mathbf{l}_0^{(1)<4>})$,

$$\mathrm{E}\left[N^2\left\{\frac{\partial^2\bar{l}}{\partial(\boldsymbol{\theta}_x)_A\partial(\boldsymbol{\theta}_x)_B}-\mathrm{E}(\cdot)\right\}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_x)_C}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_x)_D}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_x)_E}\right],\text{ and}$$

$$\mathrm{E}\left[N^2\left\{\frac{\partial^2\bar{l}}{\partial(\boldsymbol{\theta}_x)_A\partial(\boldsymbol{\theta}_x)_B}-\mathrm{E}(\cdot)\right\}\left\{\frac{\partial^2\bar{l}}{\partial(\boldsymbol{\theta}_x)_C\partial(\boldsymbol{\theta}_x)_D}-\mathrm{E}(\cdot)\right\}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_x)_E}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_x)_F}\right],$$

see (1-3) and (1-4). The remaining results are

$$\mathrm{E}_X\left[\left\{\frac{\partial^3 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B (\boldsymbol{\theta}_x)_C} - \mathrm{E}_X(\cdot)\right\}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D}\right]$$

$$= \mathrm{E}_X\left\{\frac{\partial^3 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B (\boldsymbol{\theta}_x)_C}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D}\right\}$$

in

$$\mathrm{E}\left[N^2\left\{\frac{\partial^3 \overline{l}}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B \partial(\boldsymbol{\theta}_x)_C} - \mathrm{E}(\cdot)\right\}\frac{\partial \overline{l}}{\partial(\boldsymbol{\theta}_x)_D}\frac{\partial \overline{l}}{\partial(\boldsymbol{\theta}_x)_E}\frac{\partial \overline{l}}{\partial(\boldsymbol{\theta}_x)_F}\right]$$

First, we obtain the nonzero $\frac{\partial^3 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B (\boldsymbol{\theta}_x)_C}$ as

$$\frac{\partial^3 \ln p(\mathbf{x})}{\partial\sigma_{ab}\partial\boldsymbol{\mu}\partial\boldsymbol{\mu}'} = \frac{2-\delta_{ab}}{2}\{(\boldsymbol{\Sigma}^{-1})_{\cdot a}(\boldsymbol{\Sigma}^{-1})_{b\cdot} + (\boldsymbol{\Sigma}^{-1})_{\cdot b}(\boldsymbol{\Sigma}^{-1})_{a\cdot}\},$$

$$\frac{\partial^3 \ln p(\mathbf{x})}{\partial\sigma_{ab}\partial\sigma_{cd}\,\partial\boldsymbol{\mu}} = \frac{1}{4}(2-\delta_{ab})(2-\delta_{cd})\sum^{4}\sigma^{ac}[\{\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}_b(\boldsymbol{\Sigma}^{-1})_{\cdot d}$$
$$+\{\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}_d(\boldsymbol{\Sigma}^{-1})_{\cdot b}],$$

and

$$\frac{\partial^3 \ln p(\mathbf{x})}{\partial\sigma_{ab}\partial\sigma_{cd}\,\partial\sigma_{ef}} = \frac{1}{8}(2-\delta_{ab})(2-\delta_{cd})(2-\delta_{ef})$$
$$\times\left[-\sum^{8}\sigma^{ac}\sigma^{de}\sigma^{fb} + \sum^{24}\sigma^{ac}\sigma^{de}\{\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}_f\{\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}_b\right],$$

which yield the nonzero $\mathrm{E}_X\left\{\frac{\partial^3 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B (\boldsymbol{\theta}_x)_C}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D}\right\}$ as

$$\mathrm{E}_X\left\{\frac{\partial^3 \ln p(\mathbf{x})}{\partial\sigma_{ab}\partial\sigma_{cd}\partial\boldsymbol{\mu}}\frac{\partial \ln p(\mathbf{x})}{\partial\boldsymbol{\mu}'}\right\}$$

$$= \frac{1}{4}(2-\delta_{ab})(2-\delta_{cd})\sum^{4}\sigma^{ac}\{(\boldsymbol{\Sigma}^{-1})_{\cdot d}(\boldsymbol{\Sigma}^{-1})_{b\cdot} + (\boldsymbol{\Sigma}^{-1})_{\cdot b}(\boldsymbol{\Sigma}^{-1})_{d\cdot}\},$$

$$\mathrm{E}_X\left\{\frac{\partial^3 \ln p(\mathbf{x})}{\partial\sigma_{ab}\partial\sigma_{cd}\partial\sigma_{ef}}\frac{\partial \ln p(\mathbf{x})}{\partial\sigma_{gh}}\right\}=\frac{1}{16}(2-\delta_{ab})(2-\delta_{cd})(2-\delta_{ef})(2-\delta_{gh})$$
$$\times\sum^{24}\sigma^{ac}\sigma^{de}(\sigma^{fg}\sigma^{hb}+\sigma^{fh}\sigma^{gb}).$$

(1-7) $\mathrm{E}(N^3\mathbf{l}_0^{(1)<4>})-N\sum^{3}\{\mathrm{E}(N\mathbf{l}_0^{(1)<2>})\}^{<2>}$ (for α_4)

$$\frac{N^2-N}{N}\sum^{3}\mathrm{E}_X\left\{\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_B}\right\}\mathrm{E}_X\left\{\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D}\right\}$$
$$+\mathrm{E}_X\left\{\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_B}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D}\right\}-N\sum^{3}(\boldsymbol{\Lambda}_x)_{AB}(\boldsymbol{\Lambda}_x)_{CD}$$
$$=\mathrm{E}_X\left\{\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_B}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C}\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D}\right\}-\sum^{3}(\boldsymbol{\Lambda}_x)_{AB}(\boldsymbol{\Lambda}_x)_{CD}$$

(the fourth multivariate cumulant of $\partial \ln p(\mathbf{x})/\partial(\boldsymbol{\theta}_x)_A$'s).

The nonzero expectations required for the fourth cumulant are

$$\mathrm{E}_X\left\{\frac{\partial \ln p(\mathbf{x})}{\partial\sigma_{ab}}\frac{\partial \ln p(\mathbf{x})}{\partial\sigma_{cd}}\frac{\partial \ln p(\mathbf{x})}{\partial\mu_e}\frac{\partial \ln p(\mathbf{x})}{\partial\mu_f}\right\}$$
$$=\frac{1}{4}(2-\delta_{ab})(2-\delta_{cd})\left\{\sigma^{ef}(\sigma^{ac}\sigma^{bd}+\sigma^{ad}\sigma^{bc})+\sum^{8}\sigma^{ae}\sigma^{cf}\sigma^{bd}\right\},$$

$$\mathrm{E}_X\left\{\frac{\partial \ln p(\mathbf{x})}{\partial\sigma_{ab}}\frac{\partial \ln p(\mathbf{x})}{\partial\sigma_{cd}}\frac{\partial \ln p(\mathbf{x})}{\partial\sigma_{ef}}\frac{\partial \ln p(\mathbf{x})}{\partial\sigma_{gh}}\right\}$$
$$=\frac{1}{16}(2-\delta_{ab})(2-\delta_{cd})(2-\delta_{ef})(2-\delta_{gh})$$

$$\times\left\{-3\sigma^{ab}\sigma^{cd}\sigma^{ef}\sigma^{gh}+\sum^{6}\sigma^{ab}\sigma^{cd}\,(\sigma^{ef}\sigma^{gh}+\sigma^{eg}\sigma^{fh}+\sigma^{eh}\sigma^{fg})\right.$$

$$\left.-\sum^{4}\sigma^{ab}\sum^{15}\sigma^{cd}\sigma^{ef}\sigma^{gh}+\sum^{105}\sigma^{ab}\sigma^{cd}\sigma^{ef}\sigma^{gh}\right\}$$

$$=\frac{1}{16}(2-\delta_{ab})(2-\delta_{cd})(2-\delta_{ef})(2-\delta_{gh})\left(\sum^{6\times4\times2}\sigma^{ac}\sigma^{de}\sigma^{fg}\sigma^{hb}\right.$$

$$\left.+\sum_{(ab,cd,ef,gh)}^{3}\sum_{(c,d)}^{2}\sigma^{ac}\sigma^{bd}\sum_{(g,h)}^{2}\sigma^{eg}\sigma^{fh}\right)$$

(note that $\mathrm{E}_X\left\{\frac{\partial\ln p(\mathbf{x})}{\partial\mu_a}\frac{\partial\ln p(\mathbf{x})}{\partial\mu_b}\frac{\partial\ln p(\mathbf{x})}{\partial\mu_c}\frac{\partial\ln p(\mathbf{x})}{\partial\mu_d}\right\}=\sum^{3}\sigma^{ab}\sigma^{cd}$ is nonzero, but the corresponding cumulant is zero).

(1-8) $\mathrm{E}(N^3\mathbf{l}_0^{(1)<3>}\otimes\mathbf{l}_0^{(2)})$ (for α_4)

The expectations are expressed as $\sum^{10}(\cdot)$, where each $(\cdot)$ is the product of two expectations, which was shown in (1-2), (1-3), and (1-5).

(1-9) $\mathrm{E}(N^3\mathbf{l}_0^{(1)<2>}\otimes\mathbf{l}_0^{(2)<2>})$ (for α_4)

The results are expressed as $\sum^{15}(\cdot)$, where each $(\cdot)$ is the product of three expectations, which was shown in (1-1) and (1-4).

(1-10) $\mathrm{E}(N^3\mathbf{l}_0^{(1)<3>}\otimes\mathbf{l}_0^{(3)})$ (for α_4)

The results are expressed as $\sum^{15}(\cdot)$, where each $(\cdot)$ is the product of three expectations, which was shown in (1-6).

(1-11) $\mathrm{E}\left(\frac{\partial^3\overline{l}}{\partial(\boldsymbol{\theta}_x)_A\partial(\boldsymbol{\theta}_x)_B(\boldsymbol{\theta}_x)_C}\right)$ (for α_1 and α_3)

The results are given by the Bartlett identity as follows:

$$\mathrm{E}\left(\frac{\partial^3 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B (\boldsymbol{\theta}_x)_C}\right) = -\sum^{3} \mathrm{E}\left(\frac{\partial^2 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C}\right)$$
$$-\mathrm{E}\left(\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_B} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C}\right).$$

(1-12) $\mathrm{E}\left(\dfrac{\partial^4 \bar{l}}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B (\boldsymbol{\theta}_x)_C (\boldsymbol{\theta}_x)_D}\right)$ (for $\alpha_{\Delta 2}$ and α_4)

As in (1-11), we obtain

$$\mathrm{E}\left(\frac{\partial^4 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B (\boldsymbol{\theta}_x)_C (\boldsymbol{\theta}_x)_D}\right) = -\sum^{4} \mathrm{E}\left(\frac{\partial^3 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B \partial(\boldsymbol{\theta}_x)_C} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D}\right)$$
$$-\sum^{3} \mathrm{E}\left(\frac{\partial^2 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B} \frac{\partial^2 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C \partial(\boldsymbol{\theta}_x)_D}\right) - \sum^{6} \mathrm{E}\left(\frac{\partial^2 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B}\right.$$
$$\left.\times \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D}\right) - \mathrm{E}\left(\frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_B} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D}\right).$$

(2) $\boldsymbol{\beta}$ and $\boldsymbol{\xi}$

(2-1) $\mathrm{E}(N\mathbf{l}_0^{(2)}) = N\,\mathrm{E}\left\{\mathrm{v}(\mathbf{M})' \otimes \dfrac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0{}'}, \left(\dfrac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0{}'}\right)^{<2>}\right\}'$ (for α_1)

The results are

$$
\mathrm{E}\left(\frac{\partial^2 \ln \Pr(Z\,|\,\mathbf{x})}{\partial\beta_k\partial\beta_l}\frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial\beta_m}\right)
$$

$$
=\mathrm{E}\left[\left\{\delta_{kl}\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\gamma_k\phi_k-\delta_{kl}\left(\frac{\delta_{zk}}{\pi_k^2}+\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\right)\phi_k^2\right.\right.
$$

$$
\left.\left.+\left(\frac{\delta_{zk(l+1)}}{\pi_k^2}+\frac{\delta_{z(k+1)l}}{\pi_{k+1}^2}\right)\phi_k\phi_l\right\}\left(\frac{\delta_{zm}}{\pi_m}-\frac{\delta_{z(m+1)}}{\pi_{m+1}}\right)\phi_m\right]
$$

$$
=\mathrm{E}_X\left[\left\{\delta_{kl}\left(-\frac{\gamma_k\phi_k}{\pi_k}-\frac{\phi_k^2}{\pi_k^2}\right)+\frac{\delta_{k(l+1)}}{\pi_k^2}\phi_k\,\phi_{k-1}\right\}(\delta_{mk}\phi_k-\delta_{(m+1)k}\phi_{k-1})\right.
$$

$$
\left.+\left\{\delta_{kl}\left(\frac{\gamma_k\phi_k}{\pi_{k+1}}-\frac{\phi_k^2}{\pi_{k+1}^2}\right)+\frac{\delta_{(k+1)l}}{\pi_{k+1}^2}\phi_k\,\phi_{k+1}\right\}(\delta_{m(k+1)}\phi_{k+1}-\delta_{mk}\phi_k)\right],
$$

$$
\mathrm{E}\left(\frac{\partial^2 \ln \Pr(Z\,|\,\mathbf{x})}{\partial\beta_k\partial\beta_l}\frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial\boldsymbol{\xi}}\right)
$$

$$
=\mathrm{E}\left[\left\{\delta_{kl}\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\gamma_k\phi_k-\delta_{kl}\left(\frac{\delta_{zk}}{\pi_k^2}+\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\right)\phi_k^2\right.\right.
$$

$$
\left.\left.+\left(\frac{\delta_{zk(l+1)}}{\pi_k^2}+\frac{\delta_{z(k+1)l}}{\pi_{k+1}^2}\right)\phi_k\phi_l\right\}\sum_{a=1}^{K}\mathbf{x}\frac{\delta_{za}}{\pi_a}(\phi_a-\phi_{a-1})\right]
$$

$$
=\mathrm{E}_X\left(\mathbf{x}\left[\left\{\delta_{kl}\left(-\frac{\gamma_k\phi_k}{\pi_k}-\frac{\phi_k^2}{\pi_k^2}\right)+\frac{\delta_{k(l+1)}}{\pi_k^2}\phi_k\,\phi_{k-1}\right\}(\phi_k-\phi_{k-1})\right.\right.
$$

$$
\left.\left.+\left\{\delta_{kl}\left(\frac{\gamma_k\phi_k}{\pi_{k+1}}-\frac{\phi_k^2}{\pi_{k+1}^2}\right)+\frac{\delta_{(k+1)l}}{\pi_{k+1}^2}\phi_k\,\phi_{k+1}\right\}(\phi_{k+1}-\phi_k)\right]\right),
$$

$$\mathrm{E}\left(\frac{\partial^2 \ln \Pr(Z\,|\,\mathbf{x})}{\partial \boldsymbol{\xi} \partial \beta_k}\frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \beta_l}\right)$$

$$= \mathrm{E}\left[\mathbf{x}\left\{\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\gamma_k\phi_k - \frac{\delta_{zk}}{\pi_k^2}\phi_k(\phi_k-\phi_{k-1}) + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\phi_k(\phi_{k+1}-\phi_k)\right\}\right.$$

$$\left.\times\left(\frac{\delta_{zl}}{\pi_l}-\frac{\delta_{z(l+1)}}{\pi_{l+1}}\right)\phi_l\right]$$

$$= \mathrm{E}_X\left(\mathbf{x}\left[\delta_{kl}\left\{-\frac{\gamma_k\phi_k}{\pi_k}-\frac{1}{\pi_k^2}\phi_k(\phi_k-\phi_{k-1})-\left(\frac{\gamma_k\phi_k}{\pi_{k+1}}+\frac{1}{\pi_{k+1}^2}\phi_k(\phi_{k+1}-\phi_k)\right)\right\}\phi_k\right.\right.$$

$$+\delta_{k(l+1)}\left\{\frac{\gamma_k\phi_k}{\pi_k}+\frac{1}{\pi_k^2}\phi_k(\phi_k-\phi_{k-1})\right\}\phi_{k-1}$$

$$\left.\left.+\delta_{(k+1)l}\left\{\frac{\gamma_k\phi_k}{\pi_{k+1}}+\frac{1}{\pi_{k+1}^2}\phi_k(\phi_{k+1}-\phi_k)\right\}\phi_{k+1}\right]\right),$$

$$\mathrm{E}\left(\frac{\partial^2 \ln \Pr(Z\,|\,\mathbf{x})}{\partial \boldsymbol{\xi} \partial \beta_k}\frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \boldsymbol{\xi}'}\right)$$

$$= \mathrm{E}\left[\mathbf{x}\mathbf{x}'\left\{\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\gamma_k\phi_k - \frac{\delta_{zk}}{\pi_k^2}\phi_k(\phi_k-\phi_{k-1}) + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\phi_k(\phi_{k+1}-\phi_k)\right\}\right.$$

$$\left.\times\sum_{a=1}^{K}\frac{\delta_{za}}{\pi_a}(\phi_a-\phi_{a-1})\right]$$

$$= \mathrm{E}_X\left(\mathbf{x}\mathbf{x}'\left[\left\{-\frac{\gamma_k\phi_k}{\pi_k}-\frac{1}{\pi_k^2}\phi_k(\phi_k-\phi_{k-1})\right\}(\phi_k-\phi_{k-1})\right.\right.$$

$$\left.\left.+\left\{\frac{\gamma_k\phi_k}{\pi_{k+1}}+\frac{1}{\pi_{k+1}^2}\phi_k(\phi_{k+1}-\phi_k)\right\}(\phi_{k+1}-\phi_k)\right]\right),$$

$$\mathrm{E}\left(\frac{\partial^2 \ln \Pr(Z\,|\,\mathbf{x})}{\partial \boldsymbol{\xi} \partial \boldsymbol{\xi}'} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \beta_k}\right)$$

$$= \mathrm{E}\left[\mathbf{x}\mathbf{x}' \sum_{m=1}^{K} \left\{\frac{1}{\pi_m}(-\gamma_m \phi_m + \gamma_{m-1}\phi_{m-1}) - \frac{1}{\pi_m^2}(\phi_m - \phi_{m-1})^2\right\} \times \delta_{zm}\left(\frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\phi_k\right]$$

$$= \mathrm{E}_X\left[\mathbf{x}\mathbf{x}'\left\{\frac{1}{\pi_k}(-\gamma_k\phi_k + \gamma_{k-1}\phi_{k-1}) - \frac{1}{\pi_k^2}(\phi_k - \phi_{k-1})^2 - \frac{1}{\pi_{k+1}}(-\gamma_{k+1}\phi_{k+1} + \gamma_k\phi_k) + \frac{1}{\pi_{k+1}^2}(\phi_{k+1} - \phi_k)^2\right\}\phi_k\right],$$

$$\mathrm{E}\left(\frac{\partial^2 \ln \Pr(Z\,|\,\mathbf{x})}{\partial \xi_k \partial \xi_l} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \xi_m}\right)$$

$$= \mathrm{E}\left[x_k x_l x_m \sum_{a=1}^{K}\left\{\frac{1}{\pi_a}(-\gamma_a\phi_a + \gamma_{a-1}\phi_{a-1}) - \frac{1}{\pi_a^2}(\phi_a - \phi_{a-1})^2\right\}\delta_{za} \times \sum_{b=1}^{K}\frac{\delta_{zb}}{\pi_b}(\phi_b - \phi_{b-1})\right]$$

$$= \mathrm{E}_X\left[x_k x_l x_m \sum_{a=1}^{K}\left\{\frac{1}{\pi_a}(-\gamma_a\phi_a + \gamma_{a-1}\phi_{a-1}) - \frac{1}{\pi_a^2}(\phi_a - \phi_{a-1})^2\right\}(\phi_a - \phi_{a-1})\right].$$

(2-2) $\mathrm{E}(N^2 \mathbf{l}_0^{(1)<3>})$ (for α_3)

$$\mathrm{E}\left(\frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \beta_k} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \beta_l} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \beta_m}\right)$$

$$= \mathrm{E}\left\{\left(\frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\phi_k\left(\frac{\delta_{zl}}{\pi_l} - \frac{\delta_{z(l+1)}}{\pi_{l+1}}\right)\phi_l\left(\frac{\delta_{zm}}{\pi_m} - \frac{\delta_{z(m+1)}}{\pi_{m+1}}\right)\phi_m\right\}$$

$$= \mathrm{E}_X \left\{ \delta_{klm} \left(\frac{1}{\pi_k^2} - \frac{1}{\pi_{k+1}^2} \right) \phi_k^3 - \sum_{(klm)}^{3} \delta_{(k+1)lm} \frac{1}{\pi_{k+1}^2} \phi_k \phi_{k+1}^2 \right.$$

$$\left. + \sum_{(klm)}^{3} \delta_{(k+1)(l+1)m} \frac{1}{\pi_{k+1}^2} \phi_k^2 \phi_{k+1} \right\},$$

$$\mathrm{E}\left(\frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \beta_k} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \beta_l} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \boldsymbol{\xi}} \right)$$

$$= \mathrm{E}\left\{ \mathbf{x} \left(\frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \phi_k \left(\frac{\delta_{zl}}{\pi_l} - \frac{\delta_{z(l+1)}}{\pi_{l+1}} \right) \phi_l \sum_{a=1}^{K} \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right\}$$

$$= \mathrm{E}_X \left(\mathbf{x} \left[\delta_{kl} \left\{ \frac{1}{\pi_k^2} \phi_k^2 (\phi_k - \phi_{k-1}) + \frac{1}{\pi_{k+1}^2} \phi_k^2 (\phi_{k+1} - \phi_k) \right\} \right. \right.$$

$$\left. \left. - \sum_{(kl)}^{2} \delta_{(k+1)l} \frac{1}{\pi_{k+1}^2} \phi_{k+1} \phi_k (\phi_{k+1} - \phi_k) \right] \right),$$

$$\mathrm{E}\left(\frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \beta_k} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \boldsymbol{\xi}} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \boldsymbol{\xi}'} \right)$$

$$= \mathrm{E}\left[\mathbf{x}\mathbf{x}' \left(\frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \phi_k \left\{ \sum_{a=1}^{K} \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right\}^2 \right]$$

$$= \mathrm{E}_X \left[\mathbf{x}\mathbf{x}' \left\{ \frac{1}{\pi_k^2} \phi_k (\phi_k - \phi_{k-1})^2 - \frac{1}{\pi_{k+1}^2} \phi_k (\phi_{k+1} - \phi_k)^2 \right\} \right],$$

$$\mathrm{E}\left(\frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \xi_k} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \xi_l} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \xi_m} \right)$$

$$= \mathrm{E}\left[x_k x_l x_m \left\{ \sum_{a=1}^{K} \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right\}^3 \right] = \mathrm{E}_X \left\{ x_k x_l x_m \sum_{a=1}^{K} \frac{1}{\pi_a^2} (\phi_a - \phi_{a-1})^3 \right\}.$$

(2-3) $\mathrm{E}(N^2 \mathbf{l}_0^{(1)<2>} \otimes \mathbf{l}_0^{(2)})$ (for α_3)

$$N^2\mathrm{E}\left(\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_z)_A}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_z)_B}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_z)_C}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_z)_D}\right)$$

$$=\sum^3\mathrm{E}\left(\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_A}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_B}\right)\mathrm{E}\left(\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_C}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_D}\right)$$

$$+O(N^{-1})$$

(see the expression of $-\boldsymbol{\Lambda}_z$ in 1.1 (2)),

$$N^2\mathrm{E}\left[\left\{\frac{\partial^2\bar{l}}{\partial(\boldsymbol{\theta}_z)_A\partial(\boldsymbol{\theta}_z)_B}-\mathrm{E}(\cdot)\right\}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_z)_C}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_z)_D}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_z)_E}\right]$$

$$=\sum^3\mathrm{E}\left(\frac{\partial^2\ln\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_A\partial(\boldsymbol{\theta}_z)_B}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_C}\right)\mathrm{E}\left(\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_D}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_E}\right)$$

$$+O(N^{-1})$$

(see (2-1) and the expression of $-\boldsymbol{\Lambda}_z$ in 1.1 (2)).

(2-4) $\mathrm{E}(N^2\mathbf{l}_0^{(2)}\,\mathbf{l}_0^{(2)}{}')$ (for $\alpha_{\Delta 2}$)

For $\mathrm{E}(N^2\mathbf{l}_0^{(1)<2>}\otimes\mathbf{l}_0^{(2)})$, see (2-3). Recall $\mathbf{L}_z=\dfrac{\partial^2\ln\Pr(Z\,|\,\mathbf{x})}{\partial\boldsymbol{\theta}_z\partial\boldsymbol{\theta}_z{}'}$, then the remaining results are $\mathrm{E}[\{(\mathbf{L}_z)_{AB}-(\boldsymbol{\Lambda}_z)_{AB}\}\{(\mathbf{L}_z)_{CD}-(\boldsymbol{\Lambda}_z)_{CD}\}]$ in

$$\sum^3\mathrm{E}[\{(\mathbf{L}_z)_{AB}-(\boldsymbol{\Lambda}_z)_{AB}\}\{(\mathbf{L}_z)_{CD}-(\boldsymbol{\Lambda}_z)_{CD}\}]\mathrm{E}\left(\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_E}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_F}\right).$$

We obtain

$$\mathrm{E}\left(\frac{\partial^2\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_k\partial\beta_l}\frac{\partial^2\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_m\partial\beta_n}\right)-(\boldsymbol{\Lambda}_z)_{\beta_k\beta_l}(\boldsymbol{\Lambda}_z)_{\beta_m\beta_n}$$

$$=\mathrm{E}\left[\left\{\delta_{kl}\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\gamma_k\phi_k-\delta_{kl}\left(\frac{\delta_{zk}}{\pi_k^2}+\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\right)\phi_k^2\right.\right.$$

$$+\left(\frac{\delta_{zk(l+1)}}{\pi_k^2}+\frac{\delta_{z(k+1)l}}{\pi_{k+1}^2}\right)\phi_k\phi_l\Bigg\}$$

$$\times\left\{\delta_{mn}\left(-\frac{\delta_{zm}}{\pi_m}+\frac{\delta_{z(m+1)}}{\pi_{m+1}}\right)\gamma_m\phi_m-\delta_{mn}\left(\frac{\delta_{zm}}{\pi_m^2}+\frac{\delta_{z(m+1)}}{\pi_{m+1}^2}\right)\phi_m^2\right.$$

$$\left.+\left(\frac{\delta_{zm(n+1)}}{\pi_m^2}+\frac{\delta_{z(m+1)n}}{\pi_{m+1}^2}\right)\phi_m\phi_n\right\}\Bigg]-(\mathbf{\Lambda}_z)_{\beta_k\beta_l}(\mathbf{\Lambda}_z)_{\beta_m\beta_n}$$

$$=\mathrm{E}_X\left(\delta_{klmn}\left\{\left(\gamma_k\phi_k+\frac{\phi_k^2}{\pi_k}\right)^2\frac{1}{\pi_k}+\left(\gamma_k\phi_k-\frac{\phi_k^2}{\pi_{k+1}}\right)^2\frac{1}{\pi_{k+1}}\right\}\right.$$

$$+\sum_{(kl,\,mn)}^{2}\delta_{kl(m+1)(n+1)}\left\{-\left(\gamma_k\phi_k+\frac{\phi_k^2}{\pi_k}\right)\left(\gamma_{k-1}\phi_{k-1}-\frac{\phi_{k-1}^2}{\pi_k}\right)\frac{1}{\pi_k}\right\}$$

$$+\sum_{(klmn)}^{4}\left[\delta_{klm(n+1)}\left\{-\left(\gamma_k\phi_k+\frac{\phi_k^2}{\pi_k}\right)\frac{1}{\pi_k^2}\phi_k\phi_{k-1}\right\}\right.$$

$$\left.+\delta_{(k+1)(l+1)(m+1)n}\left(\gamma_k\phi_k-\frac{\phi_k^2}{\pi_{k+1}}\right)\frac{1}{\pi_{k+1}^2}\phi_k\phi_{k+1}\right]$$

$$\left.+\sum_{(mn)}^{2}\delta_{k(l+1)m(n+1)}\frac{1}{\pi_k^3}\phi_k^2\phi_{k-1}^2+\sum_{(mn)}^{2}\delta_{(k+1)lm(n+1)}\frac{1}{\pi_{k+1}^3}\phi_k^2\phi_{k+1}^2\right)-(\mathbf{\Lambda}_z)_{\beta_k\beta_l}(\mathbf{\Lambda}_z)_{\beta_m\beta_n},$$

$$\mathrm{E}\left(\frac{\partial^2\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_k\partial\beta_l}\frac{\partial^2\ln\Pr(Z\,|\,\mathbf{x})}{\partial\boldsymbol{\xi}\partial\beta_m}\right)-(\mathbf{\Lambda}_z)_{\beta_k\beta_l}(\mathbf{\Lambda}_z)_{\xi\beta_m}$$

$$=\mathrm{E}\left(\mathbf{x}\left[\left\{\delta_{kl}\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\gamma_k\phi_k-\delta_{kl}\left(\frac{\delta_{zk}}{\pi_k^2}+\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\right)\phi_k^2\right.\right.\right.$$

$$\left.+\left(\frac{\delta_{zk(l+1)}}{\pi_k^2}+\frac{\delta_{z(k+1)l}}{\pi_{k+1}^2}\right)\phi_k\phi_l\right\}\left\{\left(-\frac{\delta_{zm}}{\pi_m}+\frac{\delta_{z(m+1)}}{\pi_{m+1}}\right)\gamma_m\phi_m-\frac{\delta_{zm}}{\pi_m^2}\phi_m(\phi_m-\phi_{m-1})\right.$$

$$+\frac{\delta_{z(m+1)}}{\pi_{m+1}^2}\phi_m(\phi_{m+1}-\phi_m)\Big\}\Big]\Big]\Big)-(\mathbf{\Lambda}_z)_{\beta_k\beta_l}(\mathbf{\Lambda}_z)_{\xi\beta_m}$$

$$=\mathrm{E}_X\left(\mathbf{x}\left[\delta_{klm}\left\{\left(\gamma_k\phi_k+\frac{\phi_k^2}{\pi_k}\right)\left(\frac{\gamma_m\phi_m}{\pi_m}+\frac{1}{\pi_m^2}\phi_m(\phi_m-\phi_{m-1})\right)\right.\right.\right.$$

$$\left.+\left(\gamma_k\phi_k-\frac{\phi_k^2}{\pi_{k+1}}\right)\left(\frac{\gamma_m\phi_m}{\pi_{m+1}}+\frac{1}{\pi_{m+1}^2}\phi_m(\phi_{m+1}-\phi_m)\right)\right\}$$

$$+\delta_{kl(m+1)}\left\{-\left(\gamma_k\phi_k+\frac{\phi_k^2}{\pi_k}\right)\left(\frac{\gamma_m\phi_m}{\pi_{m+1}}+\frac{1}{\pi_{m+1}^2}\phi_m(\phi_{m+1}-\phi_m)\right)\right\}$$

$$+\delta_{(k+1)(l+1)m}\left(-\gamma_k\phi_k+\frac{\phi_k^2}{\pi_{k+1}}\right)\left(\frac{\gamma_m\phi_m}{\pi_m}+\frac{1}{\pi_m^2}\phi_m(\phi_m-\phi_{m-1})\right)$$

$$+\sum_{(kl)}^{2}\delta_{k(l+1)m}\frac{-1}{\pi_k}\phi_k\phi_{k-1}\left\{\frac{\gamma_m\phi_m}{\pi_m}+\frac{1}{\pi_m^2}\phi_m(\phi_m-\phi_{m-1})\right\}$$

$$\left.\left.+\sum_{(kl)}^{2}\delta_{k(l+1)(m+1)}\frac{1}{\pi_k}\phi_k\phi_{k-1}\left\{\frac{\gamma_m\phi_m}{\pi_{m+1}}+\frac{1}{\pi_{m+1}^2}\phi_m(\phi_{m+1}-\phi_m)\right\}\right]\right)-(\mathbf{\Lambda}_z)_{\beta_k\beta_l}(\mathbf{\Lambda}_z)_{\xi\beta_m},$$

$$\mathrm{E}\left(\frac{\partial^2\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_k\partial\beta_l}\frac{\partial^2\ln\Pr(Z\,|\,\mathbf{x})}{\partial\boldsymbol{\xi}\partial\boldsymbol{\xi}'}\right)-(\mathbf{\Lambda}_z)_{\beta_k\beta_l}(\mathbf{\Lambda}_z)_{\xi\xi'}$$

$$=\mathrm{E}\left[\mathbf{x}\mathbf{x}'\left\{\delta_{kl}\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\gamma_k\phi_k-\delta_{kl}\left(\frac{\delta_{zk}}{\pi_k^2}+\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\right)\phi_k^2\right.\right.$$

$$\left.+\left(\frac{\delta_{zk(l+1)}}{\pi_k^2}+\frac{\delta_{z(k+1)l}}{\pi_{k+1}^2}\right)\phi_k\phi_l\right\}$$

$$\left.\times\sum_{m=1}^{K}\left\{\frac{1}{\pi_m}(-\gamma_m\phi_m+\gamma_{m-1}\phi_{m-1})-\frac{1}{\pi_m^2}(\phi_m-\phi_{m-1})^2\right\}\delta_{zm}\right]-(\mathbf{\Lambda}_z)_{\beta_k\beta_l}(\mathbf{\Lambda}_z)_{\xi\xi'}$$

$$
= \mathrm{E}_X \left(\mathbf{x}\mathbf{x}' \left[\delta_{kl} \left\{ -\left(\gamma_k \phi_k + \frac{\phi_k^2}{\pi_k} \right) \left(\frac{1}{\pi_k} (-\gamma_k \phi_k + \gamma_{k-1} \phi_{k-1}) - \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1})^2 \right) \right.\right.\right.
$$

$$
\left. + \left(\gamma_k \phi_k - \frac{\phi_k^2}{\pi_{k+1}} \right) \left(\frac{1}{\pi_{k+1}} (-\gamma_{k+1} \phi_{k+1} + \gamma_k \phi_k) - \frac{1}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k)^2 \right) \right\}
$$

$$
\left.\left. + \sum_{(kl)}^{2} \delta_{k(l+1)} \frac{1}{\pi_k} \phi_k \phi_{k-1} \left\{ \frac{1}{\pi_k} (-\gamma_k \phi_k + \gamma_{k-1} \phi_{k-1}) - \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1})^2 \right\} \right] \right)
$$

$$
-(\mathbf{\Lambda}_z)_{\beta_k \beta_l} (\mathbf{\Lambda}_z)_{\xi \xi'},
$$

$$
\mathrm{E}\left(\frac{\partial^2 \ln \Pr(Z \mid \mathbf{x})}{\partial \boldsymbol{\xi} \partial \beta_k} \frac{\partial^2 \ln \Pr(Z \mid \mathbf{x})}{\partial \boldsymbol{\xi}' \partial \beta_l} \right) - (\mathbf{\Lambda}_z)_{\xi \beta_k} (\mathbf{\Lambda}_z)_{\beta_l \xi'}
$$

$$
= \mathrm{E}\left[\mathbf{x}\mathbf{x}' \left\{ \left(-\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \gamma_k \phi_k - \frac{\delta_{zk}}{\pi_k^2} \phi_k (\phi_k - \phi_{k-1}) + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \phi_k (\phi_{k+1} - \phi_k) \right\} \right.
$$

$$
\left. \times \left\{ \left(-\frac{\delta_{zl}}{\pi_l} + \frac{\delta_{z(l+1)}}{\pi_{l+1}} \right) \gamma_l \phi_l - \frac{\delta_{zl}}{\pi_l^2} \phi_l (\phi_l - \phi_{l-1}) + \frac{\delta_{z(l+1)}}{\pi_{l+1}^2} \phi_l (\phi_{l+1} - \phi_l) \right\} \right]
$$

$$
-(\mathbf{\Lambda}_z)_{\xi \beta_k} (\mathbf{\Lambda}_z)_{\beta_l \xi'}
$$

$$
= \mathrm{E}_X \left(\mathbf{x}\mathbf{x}' \left[\delta_{kl} \left\{ \frac{1}{\pi_k} \left(\gamma_k \phi_k + \frac{1}{\pi_k} \phi_k (\phi_k - \phi_{k-1}) \right)^2 \right.\right.\right.
$$

$$
\left. + \frac{1}{\pi_{k+1}} \left(\gamma_k \phi_k + \frac{1}{\pi_{k+1}} \phi_k (\phi_{k+1} - \phi_k) \right)^2 \right\}
$$

$$
\left.\left. - \sum_{(kl)}^{2} \delta_{k(l+1)} \left\{ \gamma_k \phi_k + \frac{1}{\pi_k} \phi_k (\phi_k - \phi_{k-1}) \right\} \left\{ \frac{1}{\pi_k} \gamma_{k-1} \phi_{k-1} + \frac{1}{\pi_k^2} \phi_{k-1} (\phi_k - \phi_{k-1}) \right\} \right] \right)
$$

$$
-(\mathbf{\Lambda}_z)_{\xi \beta_k} (\mathbf{\Lambda}_z)_{\beta_l \xi'},
$$

$$\mathrm{E}\left(\frac{\partial^2 \ln \Pr(Z\,|\,\mathbf{x})}{\partial \xi_k\, \partial \beta_l}\frac{\partial^2 \ln \Pr(Z\,|\,\mathbf{x})}{\partial \boldsymbol{\xi}\, \partial \boldsymbol{\xi}'}\right) - (\boldsymbol{\Lambda}_z)_{\xi_k \beta_l}(\boldsymbol{\Lambda}_z)_{\boldsymbol{\xi}\boldsymbol{\xi}'}$$

$$= \mathrm{E}\left[x_k \mathbf{x}\mathbf{x}' \left\{ \left(-\frac{\delta_{zl}}{\pi_l} + \frac{\delta_{z(l+1)}}{\pi_{l+1}} \right) \gamma_l \phi_l - \frac{\delta_{zl}}{\pi_l^2} \phi_l (\phi_l - \phi_{l-1}) + \frac{\delta_{z(l+1)}}{\pi_{l+1}^2} \phi_l (\phi_{l+1} - \phi_l) \right\} \right.$$

$$\left. \times \sum_{m=1}^{K} \left\{ \frac{1}{\pi_m} (-\gamma_m \phi_m + \gamma_{m-1}\phi_{m-1}) - \frac{1}{\pi_m^2} (\phi_m - \phi_{m-1})^2 \right\} \delta_{zm} \right] - (\boldsymbol{\Lambda}_z)_{\xi_k \beta_l}(\boldsymbol{\Lambda}_z)_{\boldsymbol{\xi}\boldsymbol{\xi}'}$$

$$= \mathrm{E}_X \left(x_k \mathbf{x}\mathbf{x}' \left[- \left\{ \gamma_l \phi_l + \frac{1}{\pi_l} \phi_l (\phi_l - \phi_{l-1}) \right\} \right. \right.$$

$$\times \left\{ \frac{1}{\pi_l} (-\gamma_l \phi_l + \gamma_{l-1}\phi_{l-1}) - \frac{1}{\pi_l^2} (\phi_l - \phi_{l-1})^2 \right\} + \left\{ \gamma_l \phi_l + \frac{1}{\pi_{l+1}} \phi_l (\phi_{l+1} - \phi_l) \right\}$$

$$\left. \left. \times \left\{ \frac{1}{\pi_{l+1}} (-\gamma_{l+1} \phi_{l+1} + \gamma_l \phi_l) - \frac{1}{\pi_{l+1}^2} (\phi_{l+1} - \phi_l)^2 \right\} \right] \right) - (\boldsymbol{\Lambda}_z)_{\xi_k \beta_l}(\boldsymbol{\Lambda}_z)_{\boldsymbol{\xi}\boldsymbol{\xi}'},$$

$$\mathrm{E}\left(\frac{\partial^2 \ln \Pr(Z\,|\,\mathbf{x})}{\partial \xi_k\, \partial \xi_l}\frac{\partial^2 \ln \Pr(Z\,|\,\mathbf{x})}{\partial \xi_m\, \partial \xi_n}\right) - (\boldsymbol{\Lambda}_z)_{\xi_k \xi_l}(\boldsymbol{\Lambda}_z)_{\xi_m \xi_n}$$

$$= \mathrm{E}\left(x_k x_l x_m x_n \left[\sum_{a=1}^{K} \left\{ \frac{1}{\pi_a} (-\gamma_a \phi_a + \gamma_{a-1}\phi_{a-1}) - \frac{1}{\pi_a^2} (\phi_a - \phi_{a-1})^2 \right\} \delta_{za} \right]^2 \right)$$

$$- (\boldsymbol{\Lambda}_z)_{\xi_k \xi_l}(\boldsymbol{\Lambda}_z)_{\xi_m \xi_n}$$

$$= \mathrm{E}_X \left[x_k x_l x_m x_n \sum_{a=1}^{K} \frac{1}{\pi_a} \left\{ -\gamma_a \phi_a + \gamma_{a-1}\phi_{a-1} - \frac{1}{\pi_a} (\phi_a - \phi_{a-1})^2 \right\}^2 \right]$$

$$- (\boldsymbol{\Lambda}_z)_{\xi_k \xi_l}(\boldsymbol{\Lambda}_z)_{\xi_m \xi_n}$$

(2-5) $\mathrm{E}(N^2 \mathbf{l}_0^{(1)} \mathbf{l}_0^{(2)}{}')$ (for $\alpha_{\Delta 2}$)

For $\mathrm{E}(N^2 \mathbf{l}_0^{(1)<3>})$, see (2-2). The remaining results are

$$\mathrm{E}\left[\{(\mathbf{L}_z)_{AB}-(\boldsymbol{\Lambda}_z)_{AB}\}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_C}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_D}\right]$$

$$=\mathrm{E}_X\left\{(\mathbf{L}_z)_{AB}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_C}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_D}\right\}+(\boldsymbol{\Lambda}_z)_{AB}(\boldsymbol{\Lambda}_z)_{CD}.$$

We obtain the first term on the right-hand side of the above equation as follows:

$$\mathrm{E}\left(\frac{\partial^2\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_k\partial\beta_l}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_m}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_n}\right)$$

$$=\mathrm{E}\left\{\left[\delta_{kl}\left\{\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\gamma_k\phi_k-\left(\frac{\delta_{zk}}{\pi_k^2}+\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\right)\phi_k^2\right\}\right.\right.$$

$$\left.\left.+\left(\frac{\delta_{zk(l+1)}}{\pi_k^2}+\frac{\delta_{z(k+1)l}}{\pi_{k+1}^2}\right)\phi_k\phi_l\right]\left(\frac{\delta_{zm}}{\pi_m}-\frac{\delta_{z(m+1)}}{\pi_{m+1}}\right)\phi_m\left(\frac{\delta_{zn}}{\pi_n}-\frac{\delta_{z(n+1)}}{\pi_{n+1}}\right)\phi_n\right\}$$

$$=\mathrm{E}_X\left[\delta_{klmn}\left\{-\left(\gamma_k\phi_k+\frac{\phi_k^2}{\pi_k}\right)\frac{\phi_k^2}{\pi_k^2}+\left(\gamma_k\phi_k-\frac{\phi_k^2}{\pi_{k+1}}\right)\frac{\phi_k^2}{\pi_{k+1}^2}\right\}\right.$$

$$+\sum_{(mn)}^{2}\left\{\delta_{klm(n+1)}\left(\gamma_k\phi_k+\frac{\phi_k^2}{\pi_k}\right)\frac{1}{\pi_k^2}\phi_k\phi_{k-1}\right.$$

$$\left.-\delta_{(k+1)(l+1)m(n+1)}\left(\gamma_k\phi_k-\frac{\phi_k^2}{\pi_{k+1}}\right)\frac{1}{\pi_{k+1}^2}\phi_{k+1}\phi_k\right\}$$

$$+\delta_{kl(m+1)(n+1)}\left\{-\left(\gamma_k\phi_k+\frac{\phi_k^2}{\pi_k}\right)\frac{\phi_{k-1}^2}{\pi_k^2}\right\}+\delta_{(k+1)(l+1)mn}\left(\gamma_k\phi_k-\frac{\phi_k^2}{\pi_{k+1}}\right)\frac{\phi_{k+1}^2}{\pi_{k+1}^2}$$

$$+\sum_{(kl)}^{2}\left(\delta_{k(l+1)mn}\frac{1}{\pi_k^3}\phi_k^3\phi_{k-1}-\sum_{(mn)}^{2}\delta_{k(l+1)m(n+1)}\frac{1}{\pi_k^3}\phi_k^2\phi_{k-1}^2\right.$$

$$\left.\left.+\delta_{k(l+1)(m+1)(n+1)}\frac{1}{\pi_k^3}\phi_k\phi_{k-1}^3\right)\right],$$

$$\mathrm{E}\left(\frac{\partial^2 \ln \Pr(Z \mid \mathbf{x})}{\partial\beta_k \partial\beta_l} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial\beta_m} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \boldsymbol{\xi}}\right)$$

$$= \mathrm{E}\left\{\mathbf{x}\left[\delta_{kl}\left\{\left(-\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\gamma_k\phi_k - \left(\frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\right)\phi_k^2\right\}\right.\right.$$

$$\left.\left. + \left(\frac{\delta_{zk(l+1)}}{\pi_k^2} + \frac{\delta_{z(k+1)l}}{\pi_{k+1}^2}\right)\phi_k\phi_l\right]\left(\frac{\delta_{zm}}{\pi_m} - \frac{\delta_{z(m+1)}}{\pi_{m+1}}\right)\phi_m \sum_{a=1}^{K}\frac{\delta_{za}}{\pi_a}(\phi_a - \phi_{a-1})\right\}$$

$$= \mathrm{E}_X\left(\mathbf{x}\left[\delta_{klm}\left\{-\left(\gamma_k\phi_k + \frac{\phi_k^2}{\pi_k}\right)\frac{\phi_k}{\pi_k^2}(\phi_k - \phi_{k-1})\right.\right.\right.$$

$$\left. - \left(\gamma_k\phi_k - \frac{\phi_k^2}{\pi_{k+1}}\right)\frac{\phi_k}{\pi_{k+1}^2}(\phi_{k+1} - \phi_k)\right\} + \delta_{kl(m+1)}\left(\gamma_k\phi_k + \frac{\phi_k^2}{\pi_k}\right)\frac{\phi_{k-1}}{\pi_k^2}(\phi_k - \phi_{k-1})$$

$$+ \delta_{(k+1)(l+1)m}\left(\gamma_k\phi_k - \frac{\phi_k^2}{\pi_{k+1}}\right)\frac{\phi_{k+1}}{\pi_{k+1}^2}(\phi_{k+1} - \phi_k)$$

$$\left.\left. + \sum_{(kl)}^{2}\left\{\delta_{k(l+1)m}\frac{1}{\pi_k^3}\phi_k^2\phi_{k-1}(\phi_k - \phi_{k-1}) - \delta_{k(l+1)(m+1)}\frac{1}{\pi_k^3}\phi_k\phi_{k-1}^2(\phi_k - \phi_{k-1})\right\}\right]\right),$$

$$\mathrm{E}\left(\frac{\partial^2 \ln \Pr(Z \mid \mathbf{x})}{\partial\beta_k \partial\beta_l} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \boldsymbol{\xi}} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \boldsymbol{\xi}'}\right)$$

$$= \mathrm{E}\left(\mathbf{x}\mathbf{x}'\left[\delta_{kl}\left\{\left(-\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\gamma_k\phi_k - \left(\frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\right)\phi_k^2\right\}\right.\right.$$

$$\left.\left. + \left(\frac{\delta_{zk(l+1)}}{\pi_k^2} + \frac{\delta_{z(k+1)l}}{\pi_{k+1}^2}\right)\phi_k\phi_l\right]\left\{\sum_{a=1}^{K}\frac{\delta_{za}}{\pi_a}(\phi_a - \phi_{a-1})\right\}^2\right)$$

$$= \mathrm{E}_X\left(\mathbf{x}\mathbf{x}'\left[\delta_{kl}\left\{-\left(\gamma_k\phi_k+\frac{\phi_k^2}{\pi_k}\right)\frac{1}{\pi_k^2}(\phi_k-\phi_{k-1})^2\right.\right.\right.$$

$$\left.\left.\left.+\left(\gamma_k\phi_k-\frac{\phi_k^2}{\pi_{k+1}}\right)\frac{1}{\pi_{k+1}^2}(\phi_{k+1}-\phi_k)^2\right\}+\sum_{(kl)}^{2}\delta_{k(l+1)}\frac{1}{\pi_k^3}\phi_k\phi_{k-1}(\phi_k-\phi_{k-1})^2\right]\right),$$

$$\mathrm{E}\left(\frac{\partial^2\ln\Pr(\mathrm{Z}\,|\,\mathbf{x})}{\partial\boldsymbol{\xi}\partial\beta_k}\frac{\partial\ln\Pr(\mathrm{Z}\,|\,\mathbf{x})}{\partial\beta_l}\frac{\partial\ln\Pr(\mathrm{Z}\,|\,\mathbf{x})}{\partial\beta_m}\right)$$

$$= \mathrm{E}\left[\mathbf{x}\left\{\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\gamma_k\phi_k-\frac{\delta_{zk}}{\pi_k^2}\phi_k(\phi_k-\phi_{k-1})+\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\phi_k(\phi_{k+1}-\phi_k)\right\}\right.$$

$$\left.\times\left(\frac{\delta_{zl}}{\pi_l}-\frac{\delta_{z(l+1)}}{\pi_{l+1}}\right)\phi_l\left(\frac{\delta_{zm}}{\pi_m}-\frac{\delta_{z(m+1)}}{\pi_{m+1}}\right)\phi_m\right]$$

$$= \mathrm{E}_X\left(\mathbf{x}\left[\delta_{klm}\left\{-\left(\gamma_k\phi_k+\frac{1}{\pi_k}\phi_k(\phi_k-\phi_{k-1})\right)\frac{\phi_k^2}{\pi_k^2}\right.\right.\right.$$

$$\left.+\left(\gamma_k\phi_k+\frac{1}{\pi_{k+1}}\phi_k(\phi_{k+1}-\phi_k)\right)\frac{\phi_k^2}{\pi_{k+1}^2}\right\}$$

$$+\sum_{(lm)}^{2}\left[\delta_{kl(m+1)}\left\{\gamma_k\phi_k+\frac{1}{\pi_k}\phi_k(\phi_k-\phi_{k-1})\right\}\frac{1}{\pi_k^2}\phi_k\phi_{k-1}\right.$$

$$\left.-\delta_{(k+1)l(m+1)}\left\{\gamma_k\phi_k+\frac{1}{\pi_{k+1}}\phi_k(\phi_{k+1}-\phi_k)\right\}\frac{1}{\pi_{k+1}^2}\phi_{k+1}\phi_k\right]$$

$$\left.\left.-\delta_{k(l+1)(m+1)}\left\{\gamma_k\phi_k+\frac{1}{\pi_k}\phi_k(\phi_k-\phi_{k-1})\right\}\frac{\phi_{k-1}^2}{\pi_k^2}\right]\right),$$

$$\mathrm{E}\left(\frac{\partial^2 \ln \Pr(Z \mid \mathbf{x})}{\partial \boldsymbol{\xi} \partial \beta_k} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \beta_l} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \boldsymbol{\xi}'}\right)$$

$$= \mathrm{E}\left[\mathbf{x}\mathbf{x}'\left\{\left(-\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\gamma_k \phi_k - \frac{\delta_{zk}}{\pi_k^2}\phi_k(\phi_k - \phi_{k-1}) + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\phi_k(\phi_{k+1} - \phi_k)\right\}\right.$$

$$\left.\times\left(\frac{\delta_{zl}}{\pi_l} - \frac{\delta_{z(l+1)}}{\pi_{l+1}}\right)\phi_l \sum_{a=1}^{K}\frac{\delta_{za}}{\pi_a}(\phi_a - \phi_{a-1})\right]$$

$$= \mathrm{E}_X\left(\mathbf{x}\mathbf{x}'\left[\delta_{kl}\left\{-\left(\gamma_k\phi_k + \frac{1}{\pi_k}\phi_k(\phi_k - \phi_{k-1})\right)\frac{\phi_k}{\pi_k^2}(\phi_k - \phi_{k-1})\right.\right.\right.$$

$$\left.-\left(\gamma_k\phi_k + \frac{1}{\pi_{k+1}}\phi_k(\phi_{k+1} - \phi_k)\right)\frac{\phi_k}{\pi_{k+1}^2}(\phi_{k+1} - \phi_k)\right\}$$

$$+\delta_{k(l+1)}\left(\gamma_k\phi_k + \frac{1}{\pi_k}\phi_k(\phi_k - \phi_{k-1})\right)\frac{\phi_{k-1}}{\pi_k^2}(\phi_k - \phi_{k-1})$$

$$\left.\left.+\delta_{(k+1)l}\left(\gamma_k\phi_k + \frac{1}{\pi_{k+1}}\phi_k(\phi_{k+1} - \phi_k)\right)\frac{\phi_{k+1}}{\pi_{k+1}^2}(\phi_{k+1} - \phi_k)\right]\right),$$

$$\mathrm{E}\left(\frac{\partial^2 \ln \Pr(Z \mid \mathbf{x})}{\partial \boldsymbol{\xi} \partial \beta_k} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \xi_l} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \xi_m}\right)$$

$$= \mathrm{E}\left[\mathbf{x}x_l x_m\left\{\left(-\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\gamma_k \phi_k - \frac{\delta_{zk}}{\pi_k^2}\phi_k(\phi_k - \phi_{k-1}) + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\phi_k(\phi_{k+1} - \phi_k)\right\}\right.$$

$$\left.\times\left\{\sum_{a=1}^{K}\frac{\delta_{za}}{\pi_a}(\phi_a - \phi_{a-1})\right\}^2\right]$$

$$
= \mathrm{E}_X \left(\mathbf{x} x_l x_m \left[-\left\{ \gamma_k \phi_k + \frac{1}{\pi_k} \phi_k (\phi_k - \phi_{k-1}) \right\} \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1})^2 \right.\right.
$$

$$
\left.\left. + \left\{ \gamma_k \phi_k + \frac{1}{\pi_{k+1}} \phi_k (\phi_{k+1} - \phi_k) \right\} \frac{1}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k)^2 \right] \right),
$$

$$
\mathrm{E}\left(\frac{\partial^2 \ln \Pr(Z \mid \mathbf{x})}{\partial \boldsymbol{\xi} \partial \boldsymbol{\xi}'} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \beta_k} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \beta_l} \right)
$$

$$
= \mathrm{E}\left[\mathbf{x}\mathbf{x}' \sum_{a=1}^{K} \left\{ \frac{1}{\pi_a} (-\gamma_a \phi_a + \gamma_{a-1} \phi_{a-1}) - \frac{1}{\pi_a^2} (\phi_a - \phi_{a-1})^2 \right\} \delta_{za} \right.
$$

$$
\left. \times \left(\frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \phi_k \left(\frac{\delta_{zl}}{\pi_l} - \frac{\delta_{z(l+1)}}{\pi_{l+1}} \right) \phi_l \right]
$$

$$
= \mathrm{E}_X \left\{ \mathbf{x}\mathbf{x}' \left(\delta_{kl} \left[\left\{ \frac{1}{\pi_k} (-\gamma_k \phi_k + \gamma_{k-1} \phi_{k-1}) - \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1})^2 \right\} \frac{\phi_k^2}{\pi_k} \right.\right.\right.
$$

$$
\left. + \left\{ \frac{1}{\pi_{k+1}} (-\gamma_{k+1} \phi_{k+1} + \gamma_k \phi_k) - \frac{1}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k)^2 \right\} \frac{\phi_k^2}{\pi_{k+1}} \right]
$$

$$
\left.\left. - \sum_{(kl)}^{2} \delta_{k(l+1)} \left\{ \frac{1}{\pi_k} (-\gamma_k \phi_k + \gamma_{k-1} \phi_{k-1}) - \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1})^2 \right\} \frac{\phi_k \phi_{k-1}}{\pi_k} \right) \right\},
$$

$$
\mathrm{E}\left(\frac{\partial^2 \ln \Pr(Z \mid \mathbf{x})}{\partial \boldsymbol{\xi} \partial \boldsymbol{\xi}'} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \beta_k} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \xi_l} \right)
$$

$$
= \mathrm{E}\left[\mathbf{x}\mathbf{x}' x_l \sum_{a=1}^{K} \left\{ \frac{1}{\pi_a} (-\gamma_a \phi_a + \gamma_{a-1} \phi_{a-1}) - \frac{1}{\pi_a^2} (\phi_a - \phi_{a-1})^2 \right\} \delta_{za} \right.
$$

$$
\left. \times \left(\frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \phi_k \sum_{b=1}^{K} \frac{\delta_{zb}}{\pi_b} (\phi_b - \phi_{b-1}) \right]
$$

$$=\mathrm{E}_X\left(\mathbf{x}\mathbf{x}'x_l\left[\left\{\frac{1}{\pi_k}(-\gamma_k\phi_k+\gamma_{k-1}\phi_{k-1})-\frac{1}{\pi_k^2}(\phi_k-\phi_{k-1})^2\right\}\frac{\phi_k}{\pi_k}(\phi_k-\phi_{k-1})\right.\right.$$

$$\left.\left.-\left\{\frac{1}{\pi_{k+1}}(-\gamma_{k+1}\phi_{k+1}+\gamma_k\phi_k)-\frac{1}{\pi_{k+1}^2}(\phi_{k+1}-\phi_k)^2\right\}\frac{\phi_k}{\pi_{k+1}}(\phi_{k+1}-\phi_k)\right]\right),$$

$$\mathrm{E}\left(\frac{\partial^2\ln\Pr(Z\,|\,\mathbf{x})}{\partial\xi_k\;\partial\xi_l}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\xi_m}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\xi_n}\right)$$

$$=\mathrm{E}\left[x_k\,x_l\,x_m\,x_n\sum_{a=1}^{K}\left\{\frac{1}{\pi_a}(-\gamma_a\phi_a+\gamma_{a-1}\phi_{a-1})-\frac{1}{\pi_a^2}(\phi_a-\phi_{a-1})^2\right\}\delta_{za}\right.$$

$$\left.\times\left\{\sum_{b=1}^{K}\frac{\delta_{zb}}{\pi_b}(\phi_b-\phi_{b-1})\right\}^2\right]$$

$$=\mathrm{E}_X\left[x_kx_lx_mx_n\sum_{a=1}^{K}\left\{(-\gamma_a\phi_a+\gamma_{a-1}\phi_{a-1})-\frac{1}{\pi_a}(\phi_a-\phi_{a-1})^2\right\}\frac{1}{\pi_a^2}(\phi_a-\phi_{a-1})^2\right].$$

(2-6) $\mathrm{E}(N^2\mathbf{l}_0^{(1)}\,\mathbf{l}_0^{(3)}{}')$ (for $\alpha_{\Delta 2}$)

For $\mathrm{E}(N^2\mathbf{l}_0^{(1)<4>})$,

$$\mathrm{E}\left[N^2\left\{\frac{\partial^2\bar{l}}{\partial(\boldsymbol{\theta}_z)_A\partial(\boldsymbol{\theta}_z)_B}-\mathrm{E}(\cdot)\right\}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_z)_C}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_z)_D}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_z)_E}\right],$$

and $\mathrm{E}\left[N^2\left\{\frac{\partial^2\bar{l}}{\partial(\boldsymbol{\theta}_z)_A\partial(\boldsymbol{\theta}_z)_B}-\mathrm{E}(\cdot)\right\}\left\{\frac{\partial^2\bar{l}}{\partial(\boldsymbol{\theta}_z)_C\partial(\boldsymbol{\theta}_z)_D}-\mathrm{E}(\cdot)\right\}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_z)_E}\frac{\partial\bar{l}}{\partial(\boldsymbol{\theta}_z)_F}\right]$

see (2-3) an (2.4).

The remaining results are

$$\mathrm{E}\left[N\left\{\frac{\partial^3\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_A\partial(\boldsymbol{\theta}_z)_B\partial(\boldsymbol{\theta}_z)_C}-\mathrm{E}(\cdot)\right\}\frac{\partial\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_D}\right]$$

$$=\mathrm{E}\left\{\frac{\partial^3\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_A\partial(\boldsymbol{\theta}_z)_B\partial(\boldsymbol{\theta}_z)_C}\frac{\partial\Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_D}\right\}$$

in $\mathrm{E}\left[N\left\{\frac{\partial^3\overline{l}}{\partial(\boldsymbol{\theta}_z)_A\partial(\boldsymbol{\theta}_z)_B\partial(\boldsymbol{\theta}_z)_C}-\mathrm{E}(\cdot)\right\}\frac{\partial\overline{l}}{\partial(\boldsymbol{\theta}_z)_D}\frac{\partial\overline{l}}{\partial(\boldsymbol{\theta}_z)_E}\frac{\partial\overline{l}}{\partial(\boldsymbol{\theta}_z)_F}\right]$.

First, we derive $\frac{\partial^3 \Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_A\partial(\boldsymbol{\theta}_z)_B\partial(\boldsymbol{\theta}_z)_C}$ as follows:

$$\frac{\partial^3 \ln \Pr(Z\,|\,\mathbf{x})}{\partial\beta_k\partial\beta_l\partial\beta_m}=\delta_{klm}\left\{\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)(1-\gamma_k^2)\phi_k\right.$$
$$\left.+3\left(\frac{\delta_{zk}}{\pi_k^2}+\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\right)\gamma_k\phi_k^2+2\left(\frac{\delta_{zk}}{\pi_k^3}-\frac{\delta_{z(k+1)}}{\pi_{k+1}^3}\right)\phi_k^3\right\}$$
$$+\sum_{(klm)}^{3}\left\{\delta_{z(k+1)lm}\left(-\frac{1}{\pi_{k+1}^2}\gamma_{k+1}\phi_{k+1}\phi_k-\frac{2}{\pi_{k+1}^3}\phi_{k+1}^2\phi_k\right)\right.$$
$$\left.+\delta_{z(k+1)(l+1)m}\left(-\frac{1}{\pi_{k+1}^2}\gamma_k\phi_k\phi_{k+1}+\frac{2}{\pi_{k+1}^3}\phi_k^2\phi_{k+1}\right)\right\},$$

$$\frac{\partial^3 \ln \Pr(Z\,|\,\mathbf{x})}{\partial\beta_k\partial\beta_l\partial\boldsymbol{\xi}}=\mathbf{x}\left[\delta_{kl}\left\{\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)(1-\gamma_k^2)\phi_k\right.\right.$$
$$+\left(\frac{\delta_{zk}}{\pi_k^2}(\phi_k-\phi_{k-1})-\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}(\phi_{k+1}-\phi_k)\right)\gamma_k\phi_k$$
$$\left.+2\left(\frac{\delta_{zk}}{\pi_k^2}+\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\right)\gamma_k\phi_k^2+2\left(\frac{\delta_{zk}}{\pi_k^3}(\phi_k-\phi_{k-1})+\frac{\delta_{z(k+1)}}{\pi_{k+1}^3}(\phi_{k+1}-\phi_k)\right)\phi_k^2\right\}$$
$$-\left(\frac{\delta_{zk(l+1)}}{\pi_k^2}+\frac{\delta_{z(k+1)l}}{\pi_{k+1}^2}\right)(\gamma_k+\gamma_l)\phi_k\phi_l$$
$$\left.-2\left(\frac{\delta_{zk(l+1)}}{\pi_k^3}(\phi_k-\phi_{k-1})+\frac{\delta_{z(k+1)l}}{\pi_{k+1}^3}(\phi_{k+1}-\phi_k)\right)\phi_k\phi_l\right],$$

$$\frac{\partial^3 \ln \Pr(Z\,|\,\mathbf{x})}{\partial\beta_k \partial\boldsymbol{\xi}\partial\boldsymbol{\xi}'} = \mathbf{x}\mathbf{x}'\Bigg[\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)(1-\gamma_k^2)\phi_k$$

$$+\left\{\frac{\delta_{zk}}{\pi_k^2}(\phi_k-\phi_{k-1})-\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}(\phi_{k+1}-\phi_k)\right\}\gamma_k\phi_k$$

$$+\frac{\delta_{zk}}{\pi_k^2}\{2\gamma_k\phi_k^2-(\gamma_k+\gamma_{k-1})\phi_k\phi_{k-1}\}+\frac{\delta_{zk}}{\pi_k^3}2\phi_k(\phi_k-\phi_{k-1})^2$$

$$+\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\{2\gamma_k\phi_k^2-(\gamma_{k+1}+\gamma_k)\phi_{k+1}\phi_k\}-\frac{\delta_{z(k+1)}}{\pi_{k+1}^3}2\phi_k(\phi_{k+1}-\phi_k)^2\Bigg],$$

$$\frac{\partial^3 \ln \Pr(Z\,|\,\mathbf{x})}{\partial\xi_k\partial\xi_l\partial\xi_m} = x_k\, x_l\, x_m\sum_{a=1}^{K}\Bigg[\frac{1}{\pi_a}\{(-1+\gamma_a^2)\phi_a+(1-\gamma_{a-1}^2)\phi_{a-1}\}$$

$$+\frac{3}{\pi_a^2}(\gamma_a\phi_a-\gamma_{a-1}\phi_{a-1})(\phi_a-\phi_{a-1})+\frac{2}{\pi_a^3}(\phi_a-\phi_{a-1})^3\Bigg]\delta_{za}.$$

Then, the expectations are

$$\mathrm{E}\left(\frac{\partial^3 \ln \Pr(Z\,|\,\mathbf{x})}{\partial\beta_k\partial\beta_l\partial\beta_m}\frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial\beta_n}\right) = \mathrm{E}\Bigg(\Bigg[\delta_{klm}\Bigg\{\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)(1-\gamma_k^2)\phi_k$$

$$+3\left(\frac{\delta_{zk}}{\pi_k^2}+\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\right)\gamma_k\phi_k^2+2\left(\frac{\delta_{zk}}{\pi_k^3}-\frac{\delta_{z(k+1)}}{\pi_{k+1}^3}\right)\phi_k^3\Bigg\}$$

$$+\sum_{(klm)}^{3}\Bigg\{\delta_{z(k+1)lm}\left(-\frac{1}{\pi_{k+1}^2}\gamma_{k+1}\phi_{k+1}\phi_k-\frac{2}{\pi_{k+1}^3}\phi_{k+1}^2\phi_k\right)$$

$$+\delta_{z(k+1)(l+1)m}\left(-\frac{1}{\pi_{k+1}^2}\gamma_k\phi_k\phi_{k+1}+\frac{2}{\pi_{k+1}^3}\phi_k^2\phi_{k+1}\right)\Bigg\}\Bigg]\left(\frac{\delta_{zn}}{\pi_n}-\frac{\delta_{z(n+1)}}{\pi_{n+1}}\right)\phi_n\Bigg)$$

$$
\begin{aligned}
&= \mathrm{E}_X \left[\delta_{klmn} \left\{ -\left(\frac{1}{\pi_k} + \frac{1}{\pi_{k+1}} \right)(1-\gamma_k^2)\phi_k^2 + 3\left(\frac{1}{\pi_k^2} - \frac{1}{\pi_{k+1}^2} \right)\gamma_k \phi_k^3 \right.\right. \\
&\qquad\qquad \left. + 2\left(\frac{1}{\pi_k^3} + \frac{1}{\pi_{k+1}^3} \right)\phi_k^4 \right\} \\
&+ \sum_{(klm)}^{3} \left\{ \delta_{(k+1)lmn} \left(-\frac{1}{\pi_{k+1}^2}\gamma_{k+1}\phi_{k+1}^2\phi_k - \frac{2}{\pi_{k+1}^3}\phi_{k+1}^3\phi_k \right) \right. \\
&\qquad + \delta_{(k+1)lm(n+1)} \left(\frac{1}{\pi_{k+1}^2}\gamma_{k+1}\phi_{k+1}\phi_k^2 + \frac{2}{\pi_{k+1}^3}\phi_{k+1}^2\phi_k^2 \right) \\
&\qquad + \delta_{(k+1)(l+1)mn} \left(-\frac{1}{\pi_{k+1}^2}\gamma_k\phi_{k+1}^2\phi_k + \frac{2}{\pi_{k+1}^3}\phi_{k+1}^2\phi_k^2 \right) \\
&\qquad \left. + \delta_{(k+1)(l+1)m(n+1)} \left(\frac{1}{\pi_{k+1}^2}\gamma_k\phi_{k+1}\phi_k^2 - \frac{2}{\pi_{k+1}^3}\phi_{k+1}\phi_k^3 \right) \right\} \\
&\qquad + \delta_{klm(n+1)} \left\{ \frac{1}{\pi_k}(1-\gamma_k^2)\phi_k - \frac{3}{\pi_k^2}\gamma_k\phi_k^2 - \frac{2}{\pi_k^3}\phi_k^3 \right\}\phi_{k-1} \\
&\qquad \left. + \delta_{(k+1)(l+1)(m+1)n} \left\{ \frac{1}{\pi_{k+1}}(1-\gamma_k^2)\phi_k + \frac{3}{\pi_{k+1}^2}\gamma_k\phi_k^2 - \frac{2}{\pi_{k+1}^3}\phi_k^3 \right\}\phi_{k+1} \right],
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\left(\frac{\partial^3 \ln \Pr(Z\,|\,\mathbf{x})}{\partial\beta_k\partial\beta_l\partial\beta_m} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial\boldsymbol{\xi}} \right) &= \mathrm{E}\left\{ \mathbf{x}\left[\delta_{klm} \left\{ \left(-\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right)(1-\gamma_k^2)\phi_k \right.\right.\right. \\
&\qquad \left. + 3\left(\frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \right)\gamma_k\phi_k^2 + 2\left(\frac{\delta_{zk}}{\pi_k^3} - \frac{\delta_{z(k+1)}}{\pi_{k+1}^3} \right)\phi_k^3 \right\}
\end{aligned}
$$

$$+\sum_{(klm)}^{3}\left\{\delta_{z(k+1)lm}\left(-\frac{1}{\pi_{k+1}^2}\gamma_{k+1}\phi_{k+1}\phi_k-\frac{2}{\pi_{k+1}^3}\phi_{k+1}^2\phi_k\right)\right.$$

$$\left.\left.\left.+\delta_{z(k+1)(l+1)m}\left(-\frac{1}{\pi_{k+1}^2}\gamma_k\phi_k\phi_{k+1}+\frac{2}{\pi_{k+1}^3}\phi_k^2\phi_{k+1}\right)\right\}\right]\sum_{a=1}^{K}\frac{\delta_{za}}{\pi_a}(\phi_a-\phi_{a-1})\right\}$$

$$=\mathrm{E}_X\left\{\mathbf{x}\left(\delta_{klm}\left[\left\{-\frac{1}{\pi_k}(1-\gamma_k^2)\phi_k+\frac{3}{\pi_k^2}\gamma_k\phi_k^2+\frac{2}{\pi_k^3}\phi_k^3\right\}(\phi_k-\phi_{k-1})\right.\right.\right.$$

$$\left.+\left\{\frac{1}{\pi_{k+1}}(1-\gamma_k^2)\phi_k+\frac{3}{\pi_{k+1}^2}\gamma_k\phi_k^2-\frac{2}{\pi_{k+1}^3}\phi_k^3\right\}(\phi_{k+1}-\phi_k)\right]$$

$$+\sum_{(klm)}^{3}\left\{\delta_{(k+1)lm}\left(-\frac{1}{\pi_{k+1}^2}\gamma_{k+1}\phi_{k+1}\phi_k-\frac{2}{\pi_{k+1}^3}\phi_{k+1}^2\phi_k\right)\right.$$

$$\left.\left.\left.+\delta_{(k+1)(l+1)m}\left(-\frac{1}{\pi_{k+1}^2}\gamma_k\phi_k\phi_{k+1}+\frac{2}{\pi_{k+1}^3}\phi_k^2\phi_{k+1}\right)\right\}(\phi_{k+1}-\phi_k)\right)\right\},$$

$$\mathrm{E}\left(\frac{\partial^3\ln\Pr(Z\mid\mathbf{x})}{\partial\beta_k\partial\beta_l\partial\boldsymbol{\xi}}\frac{\partial\ln\Pr(Z\mid\mathbf{x})}{\partial\beta_m}\right)=\mathrm{E}\left\{\mathbf{x}\left[\delta_{kl}\left\{\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)(1-\gamma_k^2)\phi_k\right.\right.\right.$$

$$+\left(\frac{\delta_{zk}}{\pi_k^2}(\phi_k-\phi_{k-1})-\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}(\phi_{k+1}-\phi_k)\right)\gamma_k\phi_k+2\left(\frac{\delta_{zk}}{\pi_k^2}+\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\right)\gamma_k\phi_k^2$$

$$\left.+2\left(\frac{\delta_{zk}}{\pi_k^3}(\phi_k-\phi_{k-1})+\frac{\delta_{z(k+1)}}{\pi_{k+1}^3}(\phi_{k+1}-\phi_k)\right)\phi_k^2\right\}$$

$$\left.\left.-\sum_{(kl)}^{2}\delta_{zk(l+1)}\left\{\frac{1}{\pi_k^2}(\gamma_k+\gamma_{k-1})+\frac{2}{\pi_k^3}(\phi_k-\phi_{k-1})\right\}\phi_k\phi_{k-1}\right]\left(\frac{\delta_{zm}}{\pi_m}-\frac{\delta_{z(m+1)}}{\pi_{m+1}}\right)\phi_m\right\}$$

$$
\begin{aligned}
&=\mathrm{E}_X\left\{\mathbf{x}\left(\delta_{klm}\left\{-\left(\frac{1}{\pi_k}+\frac{1}{\pi_{k+1}}\right)(1-\gamma_k^2)\phi_k^2\right.\right.\right.\\
&\qquad+\left(\frac{1}{\pi_k^2}(\phi_k-\phi_{k-1})+\frac{1}{\pi_{k+1}^2}(\phi_{k+1}-\phi_k)\right)\gamma_k\phi_k^2+2\left(\frac{1}{\pi_k^2}-\frac{1}{\pi_{k+1}^2}\right)\gamma_k\phi_k^3\\
&\qquad\left.+2\left(\frac{1}{\pi_k^3}(\phi_k-\phi_{k-1})-\frac{1}{\pi_{k+1}^3}(\phi_{k+1}-\phi_k)\right)\phi_k^3\right\}\\
&+\sum_{(kl)}^{2}\left[-\delta_{k(l+1)m}\left\{\frac{1}{\pi_k^2}(\gamma_k+\gamma_{k-1})+\frac{2}{\pi_k^3}(\phi_k-\phi_{k-1})\right\}\phi_k^2\phi_{k-1}\right.\\
&\left.\left.\left.+\delta_{k(l+1)(m+1)}\left\{\frac{1}{\pi_k^2}(\gamma_k+\gamma_{k-1})+\frac{2}{\pi_k^3}(\phi_k-\phi_{k-1})\right\}\phi_k\phi_{k-1}^2\right]\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{E}\left(\frac{\partial^3\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_k\partial\beta_l\partial\boldsymbol{\xi}}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\boldsymbol{\xi}'}\right)=\mathrm{E}\left\{\mathbf{x}\mathbf{x}'\left[\delta_{kl}\left\{\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)(1-\gamma_k^2)\phi_k\right.\right.\right.\\
&+\left(\frac{\delta_{zk}}{\pi_k^2}(\phi_k-\phi_{k-1})-\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}(\phi_{k+1}-\phi_k)\right)\gamma_k\phi_k+2\left(\frac{\delta_{zk}}{\pi_k^2}+\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\right)\gamma_k\phi_k^2\\
&\left.+2\left(\frac{\delta_{zk}}{\pi_k^3}(\phi_k-\phi_{k-1})+\frac{\delta_{z(k+1)}}{\pi_{k+1}^3}(\phi_{k+1}-\phi_k)\right)\phi_k^2\right\}\\
&\left.\left.-\sum_{(kl)}^{2}\delta_{zk(l+1)}\left\{\frac{1}{\pi_k^2}(\gamma_k+\gamma_{k-1})+\frac{2}{\pi_k^3}(\phi_k-\phi_{k-1})\right\}\phi_k\phi_{k-1}\right]\sum_{a=1}^{K}\frac{\delta_{za}}{\pi_a}(\phi_a-\phi_{a-1})\right\}\\
&=\mathrm{E}_X\left(\mathbf{x}\mathbf{x}'\left\{\delta_{kl}\left[\left\{-\frac{1}{\pi_k}(1-\gamma_k^2)\phi_k+\frac{1}{\pi_k^2}(\phi_k-\phi_{k-1})\gamma_k\phi_k+\frac{2}{\pi_k^2}\gamma_k\phi_k^2\right.\right.\right.\right.\\
&\left.+\frac{2}{\pi_k^3}(\phi_k-\phi_{k-1})\phi_k^2\right\}(\phi_k-\phi_{k-1})+\left\{\frac{1}{\pi_{k+1}}(1-\gamma_k^2)\phi_k-\frac{1}{\pi_{k+1}^2}(\phi_{k+1}-\phi_k)\gamma_k\phi_k\right.
\end{aligned}
$$

$$+\frac{2}{\pi_{k+1}^2}\gamma_k\phi_k^2+\frac{2}{\pi_{k+1}^3}(\phi_{k+1}-\phi_k)\phi_k^2\Bigg\}(\phi_{k+1}-\phi_k)\Bigg]$$

$$-\sum_{(k\,l)}^{2}\delta_{k(l+1)}\left\{\frac{1}{\pi_k^2}(\gamma_k+\gamma_{k-1})+\frac{2}{\pi_k^3}(\phi_k-\phi_{k-1})\right\}\phi_k\phi_{k-1}(\phi_k-\phi_{k-1})\Bigg\}\Bigg)\Bigg),$$

$$\mathrm{E}\left(\frac{\partial^3\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_k\partial\boldsymbol{\xi}\partial\boldsymbol{\xi}'}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_l}\right)=\mathrm{E}\left\{\mathbf{x}\mathbf{x}'\left[\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)(1-\gamma_k^2)\phi_k\right.\right.$$

$$+\left\{\frac{\delta_{zk}}{\pi_k^2}(\phi_k-\phi_{k-1})-\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}(\phi_{k+1}-\phi_k)\right\}\gamma_k\phi_k$$

$$+\frac{\delta_{zk}}{\pi_k^2}\{2\gamma_k\phi_k^2-(\gamma_k+\gamma_{k+1})\phi_k\phi_{k-1}\}+\frac{\delta_{zk}}{\pi_k^3}2\phi_k(\phi_k-\phi_{k-1})^2$$

$$\left.+\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\{2\gamma_k\phi_k^2-(\gamma_{k+1}+\gamma_k)\phi_{k+1}\phi_k\}-\frac{\delta_{z(k+1)}}{\pi_{k+1}^3}2\phi_k(\phi_{k+1}-\phi_k)^2\right]$$

$$\left.\times\left(\frac{\delta_{zl}}{\pi_l}-\frac{\delta_{z(l+1)}}{\pi_{l+1}}\right)\phi_l\right\}$$

$$=\mathrm{E}_X\left\{\mathbf{x}\mathbf{x}'\left(\delta_{kl}\left[-\left(\frac{1}{\pi_k}+\frac{1}{\pi_{k+1}}\right)(1-\gamma_k^2)\phi_k^2\right.\right.\right.$$

$$+\left\{\frac{1}{\pi_k^2}(\phi_k-\phi_{k-1})+\frac{1}{\pi_{k+1}^2}(\phi_{k+1}-\phi_k)\right\}\gamma_k\phi_k^2$$

$$+\frac{1}{\pi_k^2}\{2\gamma_k\phi_k^3-(\gamma_k+\gamma_{k-1})\phi_k^2\phi_{k-1}\}+\frac{1}{\pi_k^3}2\phi_k^2(\phi_k-\phi_{k-1})^2$$

$$\left.-\frac{1}{\pi_{k+1}^2}\{2\gamma_k\phi_k^3-(\gamma_{k+1}+\gamma_k)\phi_{k+1}\phi_k^2\}+\frac{1}{\pi_{k+1}^3}2\phi_k^2(\phi_{k+1}-\phi_k)^2\right]$$

$$+\delta_{k(l+1)}\left[\frac{1}{\pi_k}(1-\gamma_k^2)\phi_k\phi_{k-1}-\frac{1}{\pi_k^2}(\phi_k-\phi_{k-1})\gamma_k\phi_k\phi_{k-1}\right.$$

$$\left.-\frac{1}{\pi_k^2}\{2\gamma_k\phi_k^2-(\gamma_k+\gamma_{k-1})\phi_k\phi_{k-1}\}\phi_{k-1}-\frac{2}{\pi_k^3}\phi_k\phi_{k-1}(\phi_k-\phi_{k-1})^2\right]$$

$$+\delta_{(k+1)l}\left[\frac{1}{\pi_{k+1}}(1-\gamma_k^2)\phi_k\phi_{k+1}-\frac{1}{\pi_{k+1}^2}(\phi_{k+1}-\phi_k)\gamma_k\phi_k\phi_{k+1}\right.$$

$$\left.\left.\left.+\frac{1}{\pi_{k+1}^2}\{2\gamma_k\phi_k^2-(\gamma_{k+1}+\gamma_k)\phi_{k+1}\phi_k\}\phi_{k+1}-\frac{2}{\pi_{k+1}^3}\phi_k\phi_{k+1}(\phi_{k+1}-\phi_k)^2\right]\right)\right\},$$

$$\mathrm{E}\left(\frac{\partial^3\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_k\partial\boldsymbol{\xi}\partial\boldsymbol{\xi}'}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\xi_l}\right)=\mathrm{E}\left\{\mathbf{x}\mathbf{x}'x_l\left[\left(-\frac{\delta_{zk}}{\pi_k}+\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)(1-\gamma_k^2)\phi_k\right.\right.$$

$$+\left\{\frac{\delta_{zk}}{\pi_k^2}(\phi_k-\phi_{k-1})-\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}(\phi_{k+1}-\phi_k)\right\}\gamma_k\phi_k$$

$$+\frac{\delta_{zk}}{\pi_k^2}\{2\gamma_k\phi_k^2-(\gamma_k+\gamma_{k-1})\phi_k\phi_{k-1}\}+\frac{\delta_{zk}}{\pi_k^3}2\phi_k(\phi_k-\phi_{k-1})^2$$

$$\left.+\frac{\delta_{z(k+1)}}{\pi_{k+1}^2}\{2\gamma_k\phi_k^2-(\gamma_{k+1}+\gamma_k)\phi_{k+1}\phi_k\}-\frac{\delta_{z(k+1)}}{\pi_{k+1}^3}2\phi_k(\phi_{k+1}-\phi_k)^2\right]$$

$$\left.\times\sum_{a=1}^{K}\frac{\delta_{za}}{\pi_a}(\phi_a-\phi_{a-1})\right\}$$

$$=\mathrm{E}_X\left(\mathbf{x}\mathbf{x}'x_l\left\{\left[-\frac{1}{\pi_k}(1-\gamma_k^2)\phi_k+\frac{1}{\pi_k^2}(\phi_k-\phi_{k-1})\gamma_k\phi_k\right.\right.\right.$$

$$\left.+\frac{1}{\pi_k^2}\{2\gamma_k\phi_k^2-(\gamma_k+\gamma_{k-1})\phi_k\phi_{k-1}\}+\frac{2}{\pi_k^3}\phi_k(\phi_k-\phi_{k-1})^2\right](\phi_k-\phi_{k-1})$$

$$+\left[\frac{1}{\pi_{k+1}}(1-\gamma_k^2)\phi_k-\frac{1}{\pi_{k+1}^2}(\phi_{k+1}-\phi_k)\gamma_k\phi_k\right.$$

$$\left.+\frac{1}{\pi_{k+1}^2}\{2\gamma_k\phi_k^2-(\gamma_{k+1}+\gamma_k)\phi_{k+1}\phi_k\}-\frac{2}{\pi_{k+1}^3}\phi_k(\phi_{k+1}-\phi_k)^2\right](\phi_{k+1}-\phi_k)\Bigg\}\Bigg),$$

$$\mathrm{E}\left(\frac{\partial^3\ln\Pr(Z\,|\,\mathbf{x})}{\partial\xi_k\partial\xi_l\partial\xi_m}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_n}\right)$$

$$=\mathrm{E}\left\{x_k\,x_l\,x_m\sum_{a=1}^{K}\left[\frac{1}{\pi_a}\{(-1+\gamma_a^2)\phi_a+(1-\gamma_{a-1}^2)\phi_{a-1}\}\right.\right.$$

$$\left.\left.+\frac{3}{\pi_a^2}(\gamma_a\phi_a-\gamma_{a-1}\phi_{a-1})(\phi_a-\phi_{a-1})+\frac{2}{\pi_a^3}(\phi_a-\phi_{a-1})^3\right]\delta_{za}\left(\frac{\delta_{zn}}{\pi_n}-\frac{\delta_{z(n+1)}}{\pi_{n+1}}\right)\phi_n\right\}$$

$$=\mathrm{E}_X\left(x_k\,x_l\,x_m\left[\frac{1}{\pi_n}\{(-1+\gamma_n^2)\phi_n+(1-\gamma_{n-1}^2)\phi_{n-1}\}\phi_n\right.\right.$$

$$+\frac{3}{\pi_n^2}(\gamma_n\phi_n-\gamma_{n-1}\phi_{n-1})(\phi_n-\phi_{n-1})\phi_n+\frac{2}{\pi_n^3}(\phi_n-\phi_{n-1})^3\phi_n$$

$$-\frac{1}{\pi_{n+1}}\{(-1+\gamma_{n+1}^2)\phi_{n+1}+(1-\gamma_n^2)\phi_n\}\phi_n$$

$$\left.\left.-\frac{3}{\pi_{n+1}^2}(\gamma_{n+1}\phi_{n+1}-\gamma_n\phi_n)(\phi_{n+1}-\phi_n)\phi_n-\frac{2}{\pi_{n+1}^3}(\phi_{n+1}-\phi_n)^3\phi_n\right]\right),$$

$$\mathrm{E}\left(\frac{\partial^3\ln\Pr(Z\,|\,\mathbf{x})}{\partial\xi_k\partial\xi_l\partial\xi_m}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\xi_n}\right)$$

$$
= \mathrm{E}\left\{ x_k\, x_l\, x_m\, x_n \sum_{a=1}^{K} \left[\frac{1}{\pi_a} \{(-1+\gamma_a^2)\phi_a + (1-\gamma_{a-1}^2)\phi_{a-1}\} \right.\right.
$$

$$
\left.\left. + \frac{3}{\pi_a^2}(\gamma_a\phi_a - \gamma_{a-1}\phi_{a-1})(\phi_a - \phi_{a-1}) + \frac{2}{\pi_a^3}(\phi_a - \phi_{a-1})^3 \right] \delta_{za} \sum_{b=1}^{K} \frac{\delta_{zb}}{\pi_b}(\phi_b - \phi_{b-1}) \right\}
$$

$$
= \mathrm{E}_X \left\{ x_k\, x_l\, x_m\, x_n \sum_{a=1}^{K} \left[\frac{1}{\pi_a} \{(-1+\gamma_a^2)\phi_a + (1-\gamma_{a-1}^2)\phi_{a-1}\} \right.\right.
$$

$$
\left.\left. + \frac{3}{\pi_a^2}(\gamma_a\phi_a - \gamma_{a-1}\phi_{a-1})(\phi_a - \phi_{a-1}) + \frac{2}{\pi_a^3}(\phi_a - \phi_{a-1})^3 \right] (\phi_a - \phi_{a-1}) \right\}.
$$

(2-7) $\mathrm{E}(N^3 \mathbf{l}_0^{(1)<4>}) - N \sum^{3} \{\mathrm{E}(N \mathbf{l}_0^{(1)<2>})\}^{<2>}$ (for α_4)

$$
\frac{N^2 - N}{N} \sum^{3} \mathrm{E}\left\{ \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_A} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_B} \right\}
$$

$$
\times \mathrm{E}\left\{ \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_C} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_D} \right\} + \mathrm{E}\left\{ \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_A} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_B} \right.
$$

$$
\left. \times \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_C} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_D} \right\} - N \sum^{3} (\boldsymbol{\Lambda}_z)_{AB} (\boldsymbol{\Lambda}_z)_{CD}
$$

$$
= \mathrm{E}\left\{ \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_A} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_B} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_C} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial(\boldsymbol{\theta}_z)_D} \right\}
$$

$$
- \sum^{3} (\boldsymbol{\Lambda}_z)_{AB} (\boldsymbol{\Lambda}_z)_{CD}
$$

(the fourth multivariate cumulant of $\partial \ln \Pr(Z\,|\,\mathbf{x}) / \partial(\boldsymbol{\theta}_z)_A$'s).
The expectations required in the above cumulants are

$$
\mathrm{E}\left\{ \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \beta_k} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \beta_l} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \beta_m} \frac{\partial \ln \Pr(Z\,|\,\mathbf{x})}{\partial \beta_n} \right\}
$$

$$= \mathrm{E}\left\{\left(\frac{\delta_{zk}}{\pi_k}-\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\phi_k\left(\frac{\delta_{zl}}{\pi_l}-\frac{\delta_{z(l+1)}}{\pi_{l+1}}\right)\phi_l\right.$$

$$\left.\times\left(\frac{\delta_{zm}}{\pi_m}-\frac{\delta_{z(m+1)}}{\pi_{m+1}}\right)\phi_m\left(\frac{\delta_{zn}}{\pi_n}-\frac{\delta_{z(n+1)}}{\pi_{n+1}}\right)\phi_n\right\}$$

$$= \mathrm{E}_X\left[\delta_{klmn}\left(\frac{1}{\pi_k^3}+\frac{1}{\pi_{k+1}^3}\right)\phi_k^4-\sum_{(klmn)}^{4}\left\{\delta_{klm(n+1)}\frac{1}{\pi_k^3}\phi_k^3\phi_{k-1}\right.\right.$$

$$\left.\left.+\delta_{(k+1)(l+1)(m+1)n}\frac{1}{\pi_{k+1}^3}\phi_{k+1}\phi_k^3\right\}+\sum_{(klmn)}^{6}\delta_{kl(m+1)(n+1)}\frac{1}{\pi_k^3}\phi_k^2\phi_{k-1}^2\right],$$

$$\mathrm{E}\left\{\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_k}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_l}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_m}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\boldsymbol{\xi}}\right\}$$

$$= \mathrm{E}\left\{\mathbf{x}\left(\frac{\delta_{zk}}{\pi_k}-\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\phi_k\left(\frac{\delta_{zl}}{\pi_l}-\frac{\delta_{z(l+1)}}{\pi_{l+1}}\right)\phi_l\right.$$

$$\left.\times\left(\frac{\delta_{zm}}{\pi_m}-\frac{\delta_{z(m+1)}}{\pi_{m+1}}\right)\phi_m\sum_{a=1}^{K}\frac{\delta_{za}}{\pi_a}(\phi_a-\phi_{a-1})\right\}$$

$$= \mathrm{E}_X\left(\mathbf{x}\left[\delta_{klm}\left\{\frac{1}{\pi_k^3}\phi_k^3(\phi_k-\phi_{k-1})-\frac{1}{\pi_{k+1}^3}\phi_k^3(\phi_{k+1}-\phi_k)\right\}\right.\right.$$

$$\left.\left.+\sum_{(klm)}^{3}\left\{-\delta_{kl(m+1)}\frac{1}{\pi_k^3}\phi_k^2\phi_{k-1}(\phi_k-\phi_{k-1})+\delta_{(k+1)(l+1)m}\frac{1}{\pi_{k+1}^3}\phi_k^2\phi_{k+1}(\phi_{k+1}-\phi_k)\right\}\right]\right),$$

$$\mathrm{E}\left\{\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_k}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\beta_l}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\boldsymbol{\xi}}\frac{\partial\ln\Pr(Z\,|\,\mathbf{x})}{\partial\boldsymbol{\xi}'}\right\}$$

$$= \mathrm{E}\left[\mathbf{x}\mathbf{x}'\left(\frac{\delta_{zk}}{\pi_k}-\frac{\delta_{z(k+1)}}{\pi_{k+1}}\right)\phi_k\left(\frac{\delta_{zl}}{\pi_l}-\frac{\delta_{z(l+1)}}{\pi_{l+1}}\right)\phi_l\left\{\sum_{a=1}^{K}\frac{\delta_{za}}{\pi_a}(\phi_a-\phi_{a-1})\right\}^2\right]$$

$$= \mathrm{E}_X \left(\mathbf{x}\mathbf{x}' \left[\delta_{kl} \left\{ \frac{1}{\pi_k^3} (\phi_k - \phi_{k-1})^2 + \frac{1}{\pi_{k+1}^3} (\phi_{k+1} - \phi_k)^2 \right\} \phi_k^2 \right. \right.$$

$$\left. \left. - \sum_{(kl)}^{2} \delta_{k(l+1)} \frac{1}{\pi_k^3} \phi_k \phi_{k-1} (\phi_k - \phi_{k-1})^2 \right] \right),$$

$$\mathrm{E}\left\{ \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \beta_k} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \xi_l} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \xi_m} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \xi_n} \right\}$$

$$= \mathrm{E}\left[x_l \, x_m \, x_n \left(\frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \phi_k \left\{ \sum_{a=1}^{K} \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right\}^3 \right]$$

$$= \mathrm{E}_X \left[x_l \, x_m \, x_n \left\{ \frac{1}{\pi_k^3} \phi_k (\phi_k - \phi_{k-1})^3 - \frac{1}{\pi_{k+1}^3} \phi_k (\phi_{k+1} - \phi_k)^3 \right\} \right],$$

$$\mathrm{E}\left\{ \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \xi_k} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \xi_l} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \xi_m} \frac{\partial \ln \Pr(Z \mid \mathbf{x})}{\partial \xi_n} \right\}$$

$$= \mathrm{E}\left[x_k \, x_l \, x_m \, x_n \left\{ \sum_{a=1}^{K} \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right\}^4 \right]$$

$$= \mathrm{E}_X \left[x_k \, x_l \, x_m \, x_n \sum_{a=1}^{K} \frac{1}{\pi_a^3} (\phi_a - \phi_{a-1})^4 \right],$$

(2-8) $\mathrm{E}(N^3 \mathbf{l}_0^{(1)<3>} \otimes \mathbf{l}_0^{(2)})$ (for α_4)

The expectations are expressed as $\sum^{10}(\cdot)$, where each $(\cdot)$ is the product of two expectations (see (2-2), (2-3), and (2-5)).

(2-9) $\mathrm{E}(N^3 \mathbf{l}_0^{(1)<2>} \otimes \mathbf{l}_0^{(2)<2>})$ (for α_4)

The results are expressed as $\sum^{15}(\cdot)$, where each $(\cdot)$ is the product of three expectations (see (2-1) and (2-4)).

(2-10) $\mathrm{E}(N^3 \mathbf{l}_0^{(1)<3>} \otimes \mathbf{l}_0^{(3)})$ (for α_4)

The results are similar to those in (2-9) (see (2-6)).

(2-11) $\mathrm{E}\left(\frac{\partial^3 \bar{l}}{\partial(\boldsymbol{\theta}_z)_A \partial(\boldsymbol{\theta}_z)_B (\boldsymbol{\theta}_z)_C}\right)$ (for α_1 and α_3)

The results are given by the Bartlett identity (see (1-11)).

(2-12) $\mathrm{E}\left(\frac{\partial^4 \bar{l}}{\partial(\boldsymbol{\theta}_z)_A \partial(\boldsymbol{\theta}_z)_B (\boldsymbol{\theta}_z)_C (\boldsymbol{\theta}_z)_D}\right)$ (for $\alpha_{\Delta 2}$ and α_4)

The results are given by the Bartlett identity (see (1-12)).

(3) $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, $\boldsymbol{\beta}$, and $\boldsymbol{\xi}$

Some of the expectations of the log likelihood derivatives with respect to $\boldsymbol{\theta}_x$ and $\boldsymbol{\theta}_z$, e.g., $N^2\mathrm{E}\left(\frac{\partial \bar{l}}{\partial(\boldsymbol{\theta}_x)_A}\frac{\partial \bar{l}}{\partial(\boldsymbol{\theta}_x)_B}\frac{\partial \bar{l}}{\partial(\boldsymbol{\theta}_z)_C}\frac{\partial \bar{l}}{\partial(\boldsymbol{\theta}_z)_D}\right)$ are nonzero though the corresponding fourth cumulants are zero. The results are given as in (1-3), (1-4), (2-3), and (2-4).

2. The nonzero partial derivatives of $\boldsymbol{\eta}$ with respect to $\boldsymbol{\theta}$.

We define $S = S(\boldsymbol{\xi}, \boldsymbol{\Sigma}) = (1+\boldsymbol{\xi}'\boldsymbol{\Sigma}\boldsymbol{\xi})^{1/2}$ $(= R^{-1})$, then $\boldsymbol{\tau} = (\boldsymbol{\beta} + \mathbf{1}_{K-1}\boldsymbol{\xi}'\boldsymbol{\mu})S^{-1}$ and $\boldsymbol{\rho} = -(\mathrm{Diag}\,\boldsymbol{\Sigma})^{-1/2}\boldsymbol{\Sigma}\boldsymbol{\xi}S^{-1}$.

2.1 First derivatives

$$\frac{\partial \boldsymbol{\tau}}{\partial \mu_k} = \mathbf{1}_{K-1}\xi_k S^{-1}, \quad \frac{\partial \boldsymbol{\tau}}{\partial \sigma_{ab}} = -\boldsymbol{\tau} S^{-1}\frac{\partial S}{\partial \sigma_{ab}},$$

$$\frac{\partial \boldsymbol{\tau}}{\partial \beta_k} = \mathbf{e}_k S^{-1}, \quad \frac{\partial \boldsymbol{\tau}}{\partial \xi_k} = \mathbf{1}_{K-1}\mu_k S^{-1} - \boldsymbol{\tau} S^{-1}\frac{\partial S}{\partial \xi_k},$$

where $\mathbf{e}_k$ is the vector of an appropriate dimension whose *k*-th element is 1 and the remaining ones are 0.

$$\frac{\partial \boldsymbol{\rho}}{\partial \sigma_{ab}} = \frac{\delta_{ab}}{2}\sigma_{aa}^{-3/2}\mathbf{E}_{aa}\boldsymbol{\Sigma}\boldsymbol{\xi}\,S^{-1} - \frac{2-\delta_{ab}}{2}(\mathrm{Diag}\,\boldsymbol{\Sigma})^{-1/2}(\mathbf{E}_{ab}+\mathbf{E}_{ba})\boldsymbol{\xi}\,S^{-1}$$

$$-\boldsymbol{\rho}\,S^{-1}\frac{\partial S}{\partial \sigma_{ab}},$$

$$\frac{\partial \boldsymbol{\rho}}{\partial \xi_k} = -(\mathrm{Diag}\,\boldsymbol{\Sigma})^{-1/2}(\boldsymbol{\Sigma})_{\cdot k}S^{-1} - \boldsymbol{\rho}\,S^{-1}\frac{\partial S}{\partial \xi_k},$$

where $\frac{\partial S}{\partial \sigma_{ab}} = \frac{2-\delta_{ab}}{2}\xi_a\xi_b\,S^{-1}$, $\frac{\partial S}{\partial \xi_k} = (\boldsymbol{\Sigma}\boldsymbol{\xi})_k\,S^{-1}$, and $\mathbf{E}_{ab}$ is the matrix of an appropriate size whose (*a*, *b*)th element is 1 and the remaining ones are 0.

2.2 Second derivatives

$$\frac{\partial^2 \boldsymbol{\tau}}{\partial \sigma_{ab}\partial \mu_k} = -\mathbf{1}_{K-1}\xi_k S^{-2}\frac{\partial S}{\partial \sigma_{ab}},$$

$$\frac{\partial^2 \boldsymbol{\tau}}{\partial \sigma_{ab}\partial \sigma_{cd}} = -\sum_{(ab,cd)}^{2}\frac{\partial \boldsymbol{\tau}}{\partial \sigma_{cd}}S^{-1}\frac{\partial S}{\partial \sigma_{ab}} - \boldsymbol{\tau}\,S^{-1}\frac{\partial^2 S}{\partial \sigma_{ab}\partial \sigma_{cd}},$$

$$\frac{\partial^2 \boldsymbol{\tau}}{\partial \beta_k\partial \sigma_{ab}} = -\mathbf{e}_k S^{-2}\frac{\partial S}{\partial \sigma_{ab}},\quad \frac{\partial^2 \boldsymbol{\tau}}{\partial \xi_k\partial \mu_l} = \mathbf{1}_{K-1}\left(\delta_{kl}S^{-1} - \xi_l S^{-2}\frac{\partial S}{\partial \xi_k}\right),$$

$$\frac{\partial^2 \boldsymbol{\tau}}{\partial \xi_k\partial \sigma_{ab}} = -\frac{\partial \boldsymbol{\tau}}{\partial \xi_k}S^{-1}\frac{\partial S}{\partial \sigma_{ab}} - \frac{\partial \boldsymbol{\tau}}{\partial \sigma_{ab}}S^{-1}\frac{\partial S}{\partial \xi_k} - \boldsymbol{\tau}\,S^{-1}\frac{\partial^2 S}{\partial \xi_k\partial \sigma_{ab}},$$

$$\frac{\partial^2 \boldsymbol{\tau}}{\partial \xi_k\partial \beta_l} = -\frac{\partial \boldsymbol{\tau}}{\partial \beta_l}S^{-1}\frac{\partial S}{\partial \xi_k},\quad \frac{\partial^2 \boldsymbol{\tau}}{\partial \xi_k\partial \xi_l} = -\sum_{(kl)}^{2}\frac{\partial \boldsymbol{\tau}}{\partial \xi_l}S^{-1}\frac{\partial S}{\partial \xi_k} - \boldsymbol{\tau}\,S^{-1}\frac{\partial^2 S}{\partial \xi_k\partial \xi_l},$$

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial \sigma_{ab}\partial \sigma_{cd}} = -\frac{3}{4}\delta_{abcd}\sigma_{aa}^{-5/2}\mathbf{E}_{aa}\boldsymbol{\Sigma}\boldsymbol{\xi}\,S^{-1} + \sum_{(ab,cd)}^{2}\frac{\delta_{ab}}{4}\sigma_{aa}^{-3/2}(2-\delta_{cd})$$

$$\times(\delta_{ac}\mathbf{E}_{ad}+\delta_{ad}\mathbf{E}_{ac})\boldsymbol{\xi}\,S^{-1} - \sum_{(ab,cd)}^{2}\frac{\partial \boldsymbol{\rho}}{\partial \sigma_{cd}}S^{-1}\frac{\partial S}{\partial \sigma_{ab}} - \boldsymbol{\rho}\,S^{-1}\frac{\partial^2 S}{\partial \sigma_{ab}\partial \sigma_{cd}},$$

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial \xi_k \partial \sigma_{ab}} = \frac{\delta_{ab}}{2} \sigma_{aa}^{-3/2} \mathbf{E}_{aa}(\boldsymbol{\Sigma})_{\cdot k} S^{-1} - \frac{2-\delta_{ab}}{2} (\mathrm{Diag}\,\boldsymbol{\Sigma})^{-1/2} (\mathbf{e}_a \delta_{bk} + \mathbf{e}_b \delta_{ak}) S^{-1}$$

$$- \frac{\partial \boldsymbol{\rho}}{\partial \xi_k} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} - \frac{\partial \boldsymbol{\rho}}{\partial \sigma_{ab}} S^{-1} \frac{\partial S}{\partial \xi_k} - \boldsymbol{\rho} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \sigma_{ab}},$$

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial \xi_k \partial \xi_l} = -\sum_{(kl)}^{2} \frac{\partial \boldsymbol{\rho}}{\partial \xi_l} S^{-1} \frac{\partial S}{\partial \xi_k} - \boldsymbol{\rho} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \xi_l},$$

where

$$\frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{cd}} = -\frac{1}{4} (2-\delta_{ab})(2-\delta_{cd}) \xi_a \xi_b \xi_c \xi_d S^{-3},$$

$$\frac{\partial^2 S}{\partial \xi_k \partial \sigma_{ab}} = \frac{2-\delta_{ab}}{2} \left\{ (\delta_{ka} \xi_b + \delta_{kb} \xi_a) S^{-1} - \xi_a \xi_b S^{-2} \frac{\partial S}{\partial \xi_k} \right\},$$

$$\frac{\partial^2 S}{\partial \xi_k \partial \xi_l} = \sigma_{kl} S^{-1} - \frac{\partial S}{\partial \xi_k} \frac{\partial S}{\partial \xi_l} S^{-1}.$$

2.3 Third derivatives

$$\frac{\partial^3 \boldsymbol{\tau}}{\partial \sigma_{ab} \partial \sigma_{cd} \partial \mu_k} = -\sum_{(ab,cd)}^{2} \frac{\partial^2 \boldsymbol{\tau}}{\partial \sigma_{cd} \partial \mu_k} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} - \frac{\partial \boldsymbol{\tau}}{\partial \mu_k} S^{-1} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{cd}},$$

$$\frac{\partial^3 \boldsymbol{\tau}}{\partial \sigma_{ab} \partial \sigma_{cd} \partial \sigma_{ef}} = -\sum_{(ab,cd,ef)}^{3} \left(\frac{\partial^2 \boldsymbol{\tau}}{\partial \sigma_{cd} \partial \sigma_{ef}} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} + \frac{\partial \boldsymbol{\tau}}{\partial \sigma_{cd}} S^{-1} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{ef}} \right)$$

$$- \boldsymbol{\tau} S^{-1} \frac{\partial^3 S}{\partial \sigma_{ab} \partial \sigma_{cd} \partial \sigma_{ef}},$$

$$\frac{\partial^3 \boldsymbol{\tau}}{\partial \beta_k \partial \sigma_{ab} \partial \sigma_{cd}} = -\sum_{(ab,cd)}^{2} \frac{\partial^2 \boldsymbol{\tau}}{\partial \sigma_{cd} \partial \beta_k} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} - \frac{\partial \boldsymbol{\tau}}{\partial \beta_k} S^{-1} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{cd}},$$

$$\frac{\partial^3 \boldsymbol{\tau}}{\partial \xi_k \partial \sigma_{ab} \partial \mu_l} = -\frac{\partial^2 \boldsymbol{\tau}}{\partial \xi_k \partial \mu_l} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} - \frac{\partial^2 \boldsymbol{\tau}}{\partial \sigma_{ab} \partial \mu_l} S^{-1} \frac{\partial S}{\partial \xi_k} - \frac{\partial \boldsymbol{\tau}}{\partial \mu_l} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \sigma_{ab}},$$

$$\frac{\partial^3 \boldsymbol{\tau}}{\partial \xi_k \partial \sigma_{ab} \partial \sigma_{cd}} = \sum_{(ab,cd)}^{2} \left(-\frac{\partial^2 \boldsymbol{\tau}}{\partial \sigma_{cd} \partial \xi_k} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} + \frac{\partial \boldsymbol{\tau}}{\partial \sigma_{cd}} S^{-2} \frac{\partial S}{\partial \xi_k} \frac{\partial S}{\partial \sigma_{ab}} \right.$$

$$\left. - \frac{\partial \boldsymbol{\tau}}{\partial \sigma_{cd}} S^{-1} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \xi_k} \right)$$

$$- \frac{\partial \boldsymbol{\tau}}{\partial \xi_k} S^{-1} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{cd}} + \boldsymbol{\tau} S^{-2} \frac{\partial S}{\partial \xi_k} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{cd}} - \boldsymbol{\tau} S^{-1} \frac{\partial^3 S}{\partial \xi_k \partial \sigma_{ab} \partial \sigma_{cd}},$$

$$\frac{\partial^3 \boldsymbol{\tau}}{\partial \xi_k \partial \beta_l \partial \sigma_{ab}} = -\frac{\partial^2 \boldsymbol{\tau}}{\partial \xi_k \partial \beta_l} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} - \frac{\partial^2 \boldsymbol{\tau}}{\partial \beta_l \partial \sigma_{ab}} S^{-1} \frac{\partial S}{\partial \xi_k} - \frac{\partial \boldsymbol{\tau}}{\partial \beta_l} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \sigma_{ab}},$$

$$\frac{\partial^3 \boldsymbol{\tau}}{\partial \xi_k \partial \xi_l \partial \mu_m} = -\sum_{(kl)}^{2} \frac{\partial^2 \boldsymbol{\tau}}{\partial \xi_l \partial \mu_m} S^{-1} \frac{\partial S}{\partial \xi_k} - \frac{\partial \boldsymbol{\tau}}{\partial \mu_m} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \xi_l},$$

$$\frac{\partial^3 \boldsymbol{\tau}}{\partial \xi_k \partial \xi_l \partial \sigma_{ab}} = \sum_{(kl)}^{2} \left(-\frac{\partial^2 \boldsymbol{\tau}}{\partial \xi_l \partial \sigma_{ab}} S^{-1} \frac{\partial S}{\partial \xi_k} + \frac{\partial \boldsymbol{\tau}}{\partial \xi_l} S^{-2} \frac{\partial S}{\partial \sigma_{ab}} \frac{\partial S}{\partial \xi_k} \right.$$

$$\left. - \frac{\partial \boldsymbol{\tau}}{\partial \xi_l} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \sigma_{ab}} \right)$$

$$- \frac{\partial \boldsymbol{\tau}}{\partial \sigma_{ab}} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \xi_l} + \boldsymbol{\tau} S^{-2} \frac{\partial S}{\partial \sigma_{ab}} \frac{\partial^2 S}{\partial \xi_k \partial \xi_l} - \boldsymbol{\tau} S^{-1} \frac{\partial^3 S}{\partial \xi_k \partial \xi_l \partial \sigma_{ab}},$$

$$\frac{\partial^3 \boldsymbol{\tau}}{\partial \xi_k \partial \xi_l \partial \beta_m} = -\sum_{(kl)}^{2} \frac{\partial^2 \boldsymbol{\tau}}{\partial \xi_l \partial \beta_m} S^{-1} \frac{\partial S}{\partial \xi_k} - \frac{\partial \boldsymbol{\tau}}{\partial \beta_m} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \xi_l},$$

$$\frac{\partial^3 \boldsymbol{\tau}}{\partial \xi_k \partial \xi_l \partial \xi_m} = -\sum_{(klm)}^{3} \left(\frac{\partial^2 \boldsymbol{\tau}}{\partial \xi_l \partial \xi_m} S^{-1} \frac{\partial S}{\partial \xi_k} + \frac{\partial \boldsymbol{\tau}}{\partial \xi_l} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \xi_m} \right)$$

$$- \boldsymbol{\tau} S^{-1} \frac{\partial^3 S}{\partial \xi_k \partial \xi_l \partial \xi_m},$$

$$\frac{\partial^3 \boldsymbol{\rho}}{\partial \sigma_{ab} \partial \sigma_{cd} \partial \sigma_{ef}} = \frac{15}{8} \delta_{abcdef} \sigma_{aa}^{-7/2} \mathbf{E}_{aa} \boldsymbol{\Sigma} \boldsymbol{\xi} S^{-1} + \sum_{(ab,cd,ef)}^{3} \left\{ -\frac{3}{8} \delta_{abcd} \sigma_{aa}^{-5/2} (2 - \delta_{ef}) \right.$$

$$\left. \times (\delta_{ae} \mathbf{E}_{af} + \delta_{af} \mathbf{E}_{ae}) \boldsymbol{\xi} S^{-1} - \frac{\partial^2 \boldsymbol{\rho}}{\partial \sigma_{cd} \partial \sigma_{ef}} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} - \frac{\partial \boldsymbol{\rho}}{\partial \sigma_{ef}} S^{-1} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{cd}} \right\}$$

$$-\boldsymbol{\rho}\, S^{-1}\frac{\partial^3 S}{\partial\sigma_{ab}\partial\sigma_{cd}\partial\sigma_{ef}},$$

$$\frac{\partial^3\boldsymbol{\rho}}{\partial\xi_k\partial\sigma_{ab}\partial\sigma_{cd}} = -\frac{3}{4}\delta_{abcd}\sigma_{aa}^{-5/2}\mathbf{E}_{aa}\left\{(\boldsymbol{\Sigma})_{\bullet k} S^{-1} - \boldsymbol{\Sigma}\boldsymbol{\xi} S^{-2}\frac{\partial S}{\partial\xi_k}\right\} + \sum_{(ab,cd)}^{2}\frac{\sigma_{ab}}{4}\sigma_{aa}^{-3/2}$$

$$\times(2-\delta_{cd})\left\{(\delta_{ac}\delta_{dk}+\delta_{ad}\delta_{ck})\mathbf{e}_a S^{-1} - (\delta_{ac}\mathbf{E}_{ad}+\delta_{ad}\mathbf{E}_{ac})\boldsymbol{\xi} S^{-2}\frac{\partial S}{\partial\xi_k}\right\}$$

$$+\sum_{(ab,cd)}^{2}\left(-\frac{\partial^2\boldsymbol{\rho}}{\partial\xi_k\partial\sigma_{cd}}S^{-1}\frac{\partial S}{\partial\sigma_{ab}} + \frac{\partial\boldsymbol{\rho}}{\partial\sigma_{cd}}S^{-2}\frac{\partial S}{\partial\xi_k}\frac{\partial S}{\partial\sigma_{ab}} - \frac{\partial\boldsymbol{\rho}}{\partial\sigma_{cd}}S^{-1}\frac{\partial^2 S}{\partial\xi_k\partial\sigma_{ab}}\right)$$

$$-\frac{\partial\boldsymbol{\rho}}{\partial\xi_k}S^{-1}\frac{\partial^2 S}{\partial\sigma_{ab}\partial\sigma_{cd}} + \boldsymbol{\rho}\, S^{-2}\frac{\partial S}{\partial\xi_k}\frac{\partial^2 S}{\partial\sigma_{ab}\partial\sigma_{cd}} - \boldsymbol{\rho}\, S^{-1}\frac{\partial^3 S}{\partial\xi_k\partial\sigma_{ab}\partial\sigma_{cd}},$$

$$\frac{\partial^3\boldsymbol{\rho}}{\partial\xi_k\partial\xi_l\partial\sigma_{ab}} = \sum_{(kl)}^{2}\left(-\frac{\partial^2\boldsymbol{\rho}}{\partial\xi_l\partial\sigma_{ab}}S^{-1}\frac{\partial S}{\partial\xi_k} + \frac{\partial\boldsymbol{\rho}}{\partial\xi_l}S^{-2}\frac{\partial S}{\partial\sigma_{ab}}\frac{\partial S}{\partial\xi_k} - \frac{\partial\boldsymbol{\rho}}{\partial\xi_l}S^{-1}\frac{\partial^2 S}{\partial\xi_k\partial\sigma_{ab}}\right)$$

$$-\frac{\partial\boldsymbol{\rho}}{\partial\sigma_{ab}}S^{-1}\frac{\partial^2 S}{\partial\xi_k\partial\xi_l} + \boldsymbol{\rho}\, S^{-2}\frac{\partial S}{\partial\sigma_{ab}}\frac{\partial^2 S}{\partial\xi_k\partial\xi_l} - \boldsymbol{\rho}\, S^{-1}\frac{\partial^3 S}{\partial\xi_k\partial\xi_l\partial\sigma_{ab}},$$

$$\frac{\partial^3\boldsymbol{\rho}}{\partial\xi_k\partial\xi_l\partial\xi_m} = -\sum_{(klm)}^{3}\left(\frac{\partial^2\boldsymbol{\rho}}{\partial\xi_l\partial\xi_m}S^{-1}\frac{\partial S}{\partial\xi_k} + \frac{\partial\boldsymbol{\rho}}{\partial\xi_l}S^{-1}\frac{\partial^2 S}{\partial\xi_k\partial\xi_m}\right) - \boldsymbol{\rho}\, S^{-1}\frac{\partial^3 S}{\partial\xi_k\partial\xi_l\partial\xi_m},$$

where

$$\frac{\partial^3 S}{\partial\sigma_{ab}\partial\sigma_{cd}\partial\sigma_{ef}} = \frac{3}{8}(2-\delta_{ab})(2-\delta_{cd})(2-\delta_{ef})\xi_a\xi_b\xi_c\xi_d\xi_e\xi_f\, S^{-5},$$

$$\frac{\partial^3 S}{\partial\xi_k\partial\sigma_{ab}\partial\sigma_{cd}} = (2-\delta_{ab})(2-\delta_{cd})\left[\sum_{(ab,cd)}^{2}\left\{-\frac{1}{4}(\delta_{ka}\xi_b+\delta_{kb}\xi_a)\xi_c\xi_d S^{-3}\right\} + \frac{3}{4}\xi_a\xi_b\xi_c\xi_d S^{-4}\frac{\partial S}{\partial\xi_k}\right],$$

$$\frac{\partial^3 S}{\partial \xi_k \partial \xi_l \partial \sigma_{ab}} = \frac{2-\delta_{ab}}{2}(\delta_{ka}\delta_{lb} + \delta_{kb}\delta_{la})S^{-1} - \sigma_{kl}S^{-2}\frac{\partial S}{\partial \sigma_{ab}}$$
$$-\sum_{(kl)}^{2}\frac{\partial^2 S}{\partial \xi_k \partial \sigma_{ab}}\frac{\partial S}{\partial \xi_l}S^{-1} + \frac{\partial S}{\partial \xi_k}\frac{\partial S}{\partial \xi_l}S^{-2}\frac{\partial S}{\partial \sigma_{ab}},$$
$$\frac{\partial^3 S}{\partial \xi_k \partial \xi_l \partial \xi_m} = -\sum_{(klm)}^{3}\frac{\partial^2 S}{\partial \xi_k \partial \xi_m}\frac{\partial S}{\partial \xi_l}S^{-1}.$$

3. Computation by Gaussian quadrature

3.1 Univariate case

Stroud and Sechrest (1966, Table Five, pp.217-252) gave the following values of A_i and x_i:

$$\int_{-\infty}^{\infty}\exp(-x^2)f(x)dx \simeq \sum_{i=1}^{n}A_i f(x_i),$$

where *n*=2(1)64(4)96(8)136. Let $y = \sqrt{2}\,x.$ Then,

$$\int_{-\infty}^{\infty}\phi(y)f(y)dy = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp(-y^2/2)f(y)dy$$
$$= \int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\exp(-x^2)f(\sqrt{2}\,x)dx \simeq \frac{1}{\sqrt{\pi}}\sum_{i=1}^{n}A_i f(\sqrt{2}x_i).$$

The above result corresponds to Bock and Lieberman (1970, Equation (5)). Bock and Lieberman (1970, p.183) used *n*=64, where only 40 values of x_i were employed since A_i's for the remaining x_i's are $A_i < 10^{-14}$. Bock and Aitkin (1981, Table 1) reported the results using *n*=10 and 2.

3.2 Bivariate case

Suppose that $\mathbf{U} = (X, Y)' \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let the density of $\mathbf{U}$ at $\mathbf{u} = (x, y)'$ be

$$\phi_2(x, y) = \frac{1}{2\pi|\boldsymbol{\Sigma}|^{1/2}}\exp\left\{-\frac{1}{2}(\mathbf{u}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{u}-\boldsymbol{\mu})\right\},$$

where $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{yx} & \sigma_y^2 \end{pmatrix}$, $\boldsymbol{\mu} = (\mu_x, \mu_y)'$ and $\phi_2(\mathbf{U} = \mathbf{u}) = \phi_2(x, y)$ $= \phi_2(x, y, \boldsymbol{\mu}, \boldsymbol{\Sigma})$. Define $\phi_1(x, \mu, \sigma_x^2) = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma_x^2} \right\}$ (note $\phi_1(x, 0, 1) = \phi(x)$). Then, using the transformations $z_x = \frac{x - \mu_x}{\sigma_x}$ and $z_{y|x} = \frac{y - \mu_y - \{(\sigma_{xy} / \sigma_x^2)\}(x - \mu_x)}{[\sigma_y^2 - \{(\sigma_{xy})^2 / \sigma_x^2\}]^{1/2}} \equiv \frac{y - \mu_{y|x}}{\sigma_{y|x}}$, it follows that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_2(x, y) g(x, y) dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_1(x, \mu_x, \sigma_x^2)$$

$$\times \phi_1[y, \mu_y + (\sigma_{xy} / \sigma_x^2)(x - \mu_x), \sigma_y^2 - \{(\sigma_{xy})^2 / \sigma_x^2\}] g(x, y) dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(z_x) \phi(z_{y|x}) g(\mu_x + \sigma_x z_x, \mu_{y|x} + \sigma_{y|x} z_{y|x}) dz_x dz_{y|x}$$

$$\simeq \int_{-\infty}^{\infty} \phi(z_x) \frac{1}{\sqrt{\pi}} \sum_{j=1}^{n} A_j g(\mu_x + \sigma_x z_x, \mu_{y|x} + \sigma_{y|x} \sqrt{2} x_j) dz_x$$

$$\simeq \frac{1}{\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} A_i A_j g(\mu_x + \sigma_x \sqrt{2} x_i, \mu_{y|x}^* + \sigma_{y|x} \sqrt{2} x_j),$$

where $\mu_{y|x}^* = \mu_{y|\mu_x + \sigma_x \sqrt{2} x_i} = \mu_y + (\sigma_{xy} / \sigma_x) \sqrt{2} x_i$.

Part B

1. The partial derivatives of $\tilde{F}_{\text{ULS}}$ with respect to $\hat{\boldsymbol{\theta}}$

$$\frac{\partial \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i} = \text{tr}\left\{ (\hat{\boldsymbol{\Sigma}} - \mathbf{S}) \frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i} \right\}, \quad \frac{\partial^2 \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} = \text{tr}\left\{ (\hat{\boldsymbol{\Sigma}} - \mathbf{S}) \frac{\partial^2 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} + \frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i} \frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_j} \right\},$$

$$\frac{\partial^2 \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i \partial s_{ab}} = -(2 - \delta_{ab}) \left(\frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i} \right)_{ab},$$

$$\frac{\partial^3 \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} = \text{tr}\left\{ (\hat{\boldsymbol{\Sigma}} - \mathbf{S}) \frac{\partial^3 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} + \sum^{3} \frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i} \frac{\partial^2 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_j \partial \hat{\theta}_k} \right\},$$

$$\frac{\partial^3 \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial s_{ab}} = -(2 - \delta_{ab}) \left(\frac{\partial^2 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \right)_{ab},$$

$$\frac{\partial^4 \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l} = \text{tr}\left\{ (\hat{\boldsymbol{\Sigma}} - \mathbf{S}) \frac{\partial^4 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l} \right.$$
$$\left. + \sum^{4} \frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i} \frac{\partial^3 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l} + \sum^{3} \frac{\partial^2 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \frac{\partial^2 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_k \partial \hat{\theta}_l} \right\},$$

$$\frac{\partial^4 \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k \partial s_{ab}} = -(2 - \delta_{ab}) \left(\frac{\partial^3 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} \right)_{ab},$$

$$\frac{\partial^5 \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l \partial \hat{\theta}_m} = \text{tr}\left\{ (\hat{\boldsymbol{\Sigma}} - \mathbf{S}) \frac{\partial^5 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l \partial \hat{\theta}_m} \right.$$
$$\left. + \sum^{5} \frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i} \frac{\partial^4 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l \partial \hat{\theta}_m} + \sum^{10} \frac{\partial^2 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \frac{\partial^3 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_k \partial \hat{\theta}_l \partial \hat{\theta}_m} \right\},$$

$$\frac{\partial^5 \tilde{F}_{\text{ULS}}}{\partial\hat{\theta}_i \partial\hat{\theta}_j \partial\hat{\theta}_k \partial\hat{\theta}_l \partial s_{ab}} = -(2-\delta_{ab})\left(\frac{\partial^4 \hat{\boldsymbol{\Sigma}}}{\partial\hat{\theta}_i \partial\hat{\theta}_j \partial\hat{\theta}_k \partial\hat{\theta}_l}\right)_{ab}$$

$(i, j, k, l, m = 1, ..., q;\ p \geq a \geq b \geq 1).$

Using the above results with Lemma 1, we have the partial derivatives of $\hat{F}_{\text{ULS}}$ with respect to $\mathbf{s}$ as follows:

$$\frac{\partial \hat{F}_{\text{ULS}}}{\partial s_{ab}} = \text{tr}\left\{(\hat{\boldsymbol{\Sigma}}-\mathbf{S})\frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i}\right\}\frac{\partial \hat{\theta}_i}{\partial s_{ab}} - (2-\delta_{ab})(\hat{\boldsymbol{\Sigma}}-\mathbf{S})_{ab},$$

$$\frac{\partial^2 \hat{F}_{\text{ULS}}}{\partial s_{ab}\partial s_{cd}} = \text{tr}\left\{\frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i}\frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_j} + (\hat{\boldsymbol{\Sigma}}-\mathbf{S})\frac{\partial^2 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j}\right\}\frac{\partial \hat{\theta}_i}{\partial s_{ab}}\frac{\partial \hat{\theta}_j}{\partial s_{cd}}$$

$$+\,\text{tr}\left\{(\hat{\boldsymbol{\Sigma}}-\mathbf{S})\frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i}\right\}\frac{\partial^2 \hat{\theta}_i}{\partial s_{ab}\partial s_{cd}} - \sum^{2}(2-\delta_{ab})\frac{\partial \hat{\sigma}_{ab}}{\partial \hat{\theta}_i}\frac{\partial \hat{\theta}_i}{\partial s_{cd}}$$

$$+\,(2-\delta_{ab})\delta_{ac}\delta_{bd},$$

$$\frac{\partial^3 \hat{F}_{\text{ULS}}}{\partial s_{ab}\partial s_{cd}\partial s_{ef}} = \text{tr}\left\{\sum^{3}\frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i}\frac{\partial^2 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_j \partial \hat{\theta}_k} + (\hat{\boldsymbol{\Sigma}}-\mathbf{S})\frac{\partial^3 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k}\right\}\frac{\partial \hat{\theta}_i}{\partial s_{ab}}\frac{\partial \hat{\theta}_j}{\partial s_{cd}}\frac{\partial \hat{\theta}_k}{\partial s_{ef}}$$

$$+\text{tr}\left\{\frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i}\frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_j} + (\hat{\boldsymbol{\Sigma}}-\mathbf{S})\frac{\partial^2 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j}\right\}\sum^{3}\frac{\partial \hat{\theta}_i}{\partial s_{ab}}\frac{\partial^2 \hat{\theta}_j}{\partial s_{cd}\partial s_{ef}}$$

$$+\text{tr}\left\{(\hat{\boldsymbol{\Sigma}}-\mathbf{S})\frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i}\right\}\frac{\partial^3 \hat{\theta}_i}{\partial s_{ab}\partial s_{cd}\partial s_{ef}}$$

$$-\sum^{3}(2-\delta_{ab})\left(\frac{\partial^2 \hat{\sigma}_{ab}}{\partial \hat{\theta}_i \partial \hat{\theta}_j}\frac{\partial \hat{\theta}_i}{\partial s_{cd}}\frac{\partial \hat{\theta}_j}{\partial s_{ef}} + \frac{\partial \hat{\sigma}_{ab}}{\partial \hat{\theta}_i}\frac{\partial^2 \hat{\theta}_i}{\partial s_{cd}\partial s_{ef}}\right),$$

$$\frac{\partial^4 \hat{F}_{\text{ULS}}}{\partial s_{ab}\partial s_{cd}\partial s_{ef}\partial s_{gh}} = \text{tr}\left\{\sum^{3}\frac{\partial^2 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j}\frac{\partial^2 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_k \partial \hat{\theta}_l} + \sum^{4}\frac{\partial \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_i}\frac{\partial^3 \hat{\boldsymbol{\Sigma}}}{\partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l}\right.$$

$$+(\hat{\mathbf{\Sigma}}-\mathbf{S})\frac{\partial^4\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_i\partial\hat{\theta}_j\partial\hat{\theta}_k\partial\hat{\theta}_l}\Bigg\}\frac{\partial\hat{\theta}_i}{\partial s_{ab}}\frac{\partial\hat{\theta}_j}{\partial s_{cd}}\frac{\partial\hat{\theta}_k}{\partial s_{ef}}\frac{\partial\hat{\theta}_l}{\partial s_{gh}}$$

$$+\mathrm{tr}\left\{\sum^{3}\frac{\partial\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_i}\frac{\partial^2\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_j\partial\hat{\theta}_k}+(\hat{\mathbf{\Sigma}}-\mathbf{S})\frac{\partial^3\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_i\partial\hat{\theta}_j\partial\hat{\theta}_k}\right\}\sum^{6}\frac{\partial\hat{\theta}_i}{\partial s_{ab}}\frac{\partial\hat{\theta}_j}{\partial s_{cd}}\frac{\partial^2\hat{\theta}_k}{\partial s_{ef}\partial s_{gh}}$$

$$+\mathrm{tr}\left\{\sum^{3}\frac{\partial\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_i}\frac{\partial\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_j}+(\hat{\mathbf{\Sigma}}-\mathbf{S})\frac{\partial^2\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_i\partial\hat{\theta}_j}\right\}\left(\sum^{3}\frac{\partial^2\hat{\theta}_i}{\partial s_{ab}\partial s_{cd}}\frac{\partial^2\hat{\theta}_j}{\partial s_{ef}\partial s_{gh}}\right.$$

$$\left.+\sum^{4}\frac{\partial\hat{\theta}_i}{\partial s_{ab}}\frac{\partial^3\hat{\theta}_j}{\partial s_{cd}\partial s_{ef}\partial s_{gh}}\right)+\mathrm{tr}\left\{(\hat{\mathbf{\Sigma}}-\mathbf{S})\frac{\partial\hat{\mathbf{\Sigma}}}{\partial\hat{\theta}_i}\right\}\frac{\partial^4\hat{\theta}_i}{\partial s_{ab}\partial s_{cd}\partial s_{ef}\partial s_{gh}}$$

$$-\sum^{4}(2-\delta_{ab})\left(\frac{\partial^3\hat{\sigma}_{ab}}{\partial\hat{\theta}_i\partial\hat{\theta}_j\partial\hat{\theta}_k}\frac{\partial\hat{\theta}_i}{\partial s_{cd}}\frac{\partial\hat{\theta}_j}{\partial s_{ef}}\frac{\partial\hat{\theta}_k}{\partial s_{gh}}+\frac{\partial^2\hat{\sigma}_{ab}}{\partial\hat{\theta}_i\partial\hat{\theta}_j}\sum^{3}\frac{\partial\hat{\theta}_i}{\partial s_{cd}}\frac{\partial^2\hat{\theta}_j}{\partial s_{ef}\partial s_{gh}}\right.$$

$$\left.+\frac{\partial\hat{\sigma}_{ab}}{\partial\hat{\theta}_i}\frac{\partial^3\hat{\theta}_i}{\partial s_{cd}\partial s_{ef}\partial s_{gh}}\right)$$

$$(p\ge a\ge b\ge 1,\ p\ge c\ge d\ge 1,\ p\ge e\ge f\ge 1,\ p\ge g\ge h\ge 1),$$

where Einstein's summation convention is to be used only for subscripts i, j, k and l.

2. The chain rules

$$\frac{\partial\hat{g}}{\partial s_{ab}}=\frac{\partial\hat{g}}{\partial\hat{F}}\frac{\partial\hat{F}}{\partial s_{ab}},$$

$$\frac{\partial^2\hat{g}}{\partial s_{ab}\partial s_{cd}}=\frac{\partial^2\hat{g}}{\partial\hat{F}^2}\frac{\partial\hat{F}}{\partial s_{ab}}\frac{\partial\hat{F}}{\partial s_{cd}}+\frac{\partial\hat{g}}{\partial\hat{F}}\frac{\partial^2\hat{F}}{\partial s_{ab}\partial s_{cd}},$$

$$
\frac{\partial^3 \hat{g}}{\partial s_{ab}\partial s_{cd}\partial s_{ef}} = \frac{\partial^3 \hat{g}}{\partial \hat{F}^3}\frac{\partial \hat{F}}{\partial s_{ab}}\frac{\partial \hat{F}}{\partial s_{cd}}\frac{\partial \hat{F}}{\partial s_{ef}}
+ \sum^{3} \frac{\partial^2 \hat{g}}{\partial \hat{F}^2}\frac{\partial \hat{F}}{\partial s_{ab}}\frac{\partial^2 \hat{F}}{\partial s_{cd}\partial s_{ef}} + \frac{\partial \hat{g}}{\partial \hat{F}}\frac{\partial^3 \hat{F}}{\partial s_{ab}\partial s_{cd}\partial s_{ef}},
$$

$$
\frac{\partial^4 \hat{g}}{\partial s_{ab}\partial s_{cd}\partial s_{ef}\partial s_{gh}} = \frac{\partial^4 \hat{g}}{\partial \hat{F}^4}\frac{\partial \hat{F}}{\partial s_{ab}}\frac{\partial \hat{F}}{\partial s_{cd}}\frac{\partial \hat{F}}{\partial s_{ef}}\frac{\partial \hat{F}}{\partial s_{gh}}
+ \sum^{6} \frac{\partial^3 \hat{g}}{\partial \hat{F}^3}\frac{\partial \hat{F}}{\partial s_{ab}}\frac{\partial \hat{F}}{\partial s_{cd}}\frac{\partial^2 \hat{F}}{\partial s_{ef}\partial s_{gh}} + \sum^{3} \frac{\partial^2 \hat{g}}{\partial \hat{F}^2}\frac{\partial^2 \hat{F}}{\partial s_{ab}\partial s_{cd}}\frac{\partial^2 \hat{F}}{\partial s_{ef}\partial s_{gh}}
+ \sum^{4} \frac{\partial^2 \hat{g}}{\partial \hat{F}^2}\frac{\partial \hat{F}}{\partial s_{ab}}\frac{\partial^3 \hat{F}}{\partial s_{cd}\partial s_{ef}\partial s_{gh}} + \frac{\partial \hat{g}}{\partial \hat{F}}\frac{\partial^4 \hat{F}}{\partial s_{ab}\partial s_{cd}\partial s_{ef}\partial s_{gh}}
$$

$$
(p \ge a \ge b \ge 1, p \ge c \ge d \ge 1, p \ge e \ge f \ge 1, p \ge g \ge h \ge 1).
$$

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