Economic Review (Otaru University of Commerce),
Vol.70, No.4, **-**, March, 2020.

An expository supplement to the paper "The multiple Cantelli inequalities": Higher-order moments for Mardia's bivariate Pareto distribution

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This article supplements Ogasawara (2019a) with some cross moments for Mardia's (1962) bivariate Pareto distribution of type 1 and general recursive formulas for the higher-order cross moments. For undefined notations and missing references in the reference list, see Ogasawara (2019a).

S.1 Preliminary results

In this section, preliminary results including new ones are shown. The density is

$$\begin{aligned} & f_{X_1^*,\,X_2^*}(X_1^*=x_1^*, X_2^*=x_2^*) = f_{12}(x_1^*,x_2^*) = f(x_1^*,x_2^*) \\ & = \alpha(\alpha+1)(\beta_1\beta_2)^{\alpha+1}(\beta_2 x_1^* + \beta_1 x_2^* - \beta_1\beta_2)^{-(\alpha+2)} \\ & = \frac{\alpha(\alpha+1)/(\beta_1\beta_2)}{\{(x_1^*/\beta_1)+(x_2^*/\beta_2)-1\}^{\alpha+2}} \qquad \text{(S1.1)} \\ & (x_1^* \ge \beta_1 > 0,\ x_2^* \ge \beta_2 > 0,\ \alpha > 0), \end{aligned}$$

where the notation $f(x_1^*,x_2^*)$ and similar ones in this article are used for simplicity when confusion does not occur. The joint survival function is given by

$$\begin{aligned} & \Pr(\{X_1^* > x_1^*\} \cap \{X_2^* > x_2^*\}) = S_{X_1^*,\,X_2^*}(x_1^*,x_2^*) = S(x_1^*,x_2^*) \\ & = \frac{1}{\{(x_1^*/\beta_1)+(x_2^*/\beta_2)-1\}^{\alpha}}. \qquad \text{(S1.2)} \end{aligned}$$

Let $X_{*i} = X_i^*/\beta_i\ (i=1,\ 2)$ be the standardized variables with unit scale parameters i.e., $\beta_i = \beta_{*i} = 1\ (i=1,\ 2)$ (do not confuse X_{*i} with $X_i = X_i^* - \mathrm{E}(X_i^*)$ defined in Ogasawara, 2019a). Then,

$$f_{X_{*1},\,X_{*2}}(X_{*1}=x_{*1},X_{*2}=x_{*2})=f(x_{*1},x_{*2})$$
$$=\frac{\alpha(\alpha+1)}{(x_{*1}+x_{*2}-1)^{\alpha+2}}\quad(x_{*1}\ge 1,\ x_{*2}\ge 1,\ \alpha>0).\qquad\text{(S1.3)}$$

Note that X_{*1} and X_{*2} are exchangeable in that the density when x_{*1} and x_{*2} are exchanged in (S1.3) is unchanged as in the case of X_1^* and X_2^* with equal scale parameters. The survival function for X_{*1} and X_{*2} is given by

$$S_{X_{*1},\,X_{*2}}(x_{*1},x_{*2})=S(x_{*1},x_{*2})=\frac{1}{(x_{*1}+x_{*2}-1)^{\alpha}}.\qquad\text{(S1.4)}$$

The Paretian marginal distributions are

$$f_{X_{*i}}(X_{*i}=x_{*i})=f_i(x_{*i})=f(x_{*i})=\alpha/x_{*i}^{\alpha+1}\,(x_{*i}\ge 1;\ i=1,\,2;\ \alpha>0).\qquad\text{(S1.5)}$$

This univariate distribution is denoted by $P_{\mathrm{I}}(\beta,\,\alpha)$ with $\beta=\beta_{*i}=1$ for (S1.5). The survival function of $P_{\mathrm{I}}(1,\,\alpha)$ i.e., for X_{*i} is as simple as

$$S_{X_{*i}}(x_{*i})=S(x_{*i})=1/x_{*i}^{\alpha}\ \ (x_{*i}\ge 1;\ i=1,\,2).\qquad\text{(S1.6)}$$

Lemma S1.

$$\mathrm{E}_{X_{*i}}(X_{*i}^{m})=\mathrm{E}(X_{*i}^{m})=\alpha/(\alpha-m)\ \ (\alpha>m;\,i=1,\,2).\qquad\text{(S1.7)}$$

Proof 1. The direct proof is given by

$$\begin{aligned}\mathrm{E}(X_{*i}^{m})&=\int_1^{+\infty}x_{*i}^{m}(\alpha/x_{*i}^{\alpha+1})\mathrm{d}x_{*i}\\&=\{\alpha/(\alpha-m)\}\int_1^{+\infty}(\alpha-m)(1/x_{*i}^{\alpha-m+1})\mathrm{d}x_{*i}\\&=\alpha/(\alpha-m).\ \ \text{Q.E.D.}\end{aligned}\qquad\text{(S1.8)}$$

Proof 2. When m = 0, (S1.7) trivially holds. Assume that $m\ne 0$. When m = 1, it is known that since $X_{*i}\ (i=1,\,2)$ are non-negative random variables, $\mathrm{E}(X_{*i})=\int_0^{+\infty}S(x_{*i})\,\mathrm{d}x_{*i}$, which gives $\mathrm{E}(X_{*i})$ $=1+\int_1^{+\infty}(1/x_{*i}^{\alpha})\,\mathrm{d}x_{*i}=1+\{1/(\alpha-1)\}=\alpha/(\alpha-1)\ (i=1,\,2)$. Then, from (S1.6), $S(x_{*i})=1/x_{*i}^{\alpha}=1/(x_{*i}^{m})^{(\alpha/m)}$. This shows that X_{*i}^{m} follows $P_{\mathrm{I}}(1,\,\alpha/m)$ giving $\mathrm{E}(X_{*i}^{m})=(\alpha/m)/\{(\alpha/m)-1\}$.

$= \alpha / (\alpha - m)\ \ (a > m; i = 1,\ 2).$ Q.E.D.

Proof 3. The proof when m = 1 is the same as in Proof 2. When $m \neq 0$,

$$\mathrm{E}(X_{*i}^m) = \int_0^{+\infty} S_{X_{*i}^m}(x_{*i}^m)\,\mathrm{d}x_{*i}^m = \int_0^{+\infty} m x_{*i}^{m-1} S_{X_{*i}^m}(x_{*i}^m)\,\mathrm{d}x_{*i} = \int_0^{+\infty} S_{X_{*i}}(x_{*i}^{1/m})\,\mathrm{d}x_{*i}$$

$$= 1 + \int_1^{+\infty} \{1/(x_{*i}^{1/m})^{\alpha}\}\,\mathrm{d}x_{*i} = 1 + \{(\alpha/m) - 1\}^{-1} = \alpha/(\alpha - m).$$ Q.E.D.

Note that $\mathrm{E}(X_i^{*m}) = \beta_i^m \mathrm{E}(X_{*i}^m) = \beta_i^m \alpha / (\alpha - m)\ \ (i = 1, 2)$.

Lemma S2.

$$\mathrm{E}_{X_{*i}}(X_{*1}X_{*2}) = \frac{\alpha^2 - \alpha - 1}{(\alpha - 2)(\alpha - 1)}\ \ (\alpha > 2). \quad \text{(S1.9)}$$

Proof 1. This proof was used by e.g., Kotz et al. (2000, p.580) for unstandardized X_1^* and X_2^*. The proof using X_{*1} and X_{*2} is shown since the proof becomes somewhat simpler. The density function of X_{*2} given $X_{*1} = x_{*1}$ is

$$\begin{aligned} & f_{X_{*2}|X_{*1}}(X_{*2} = x_{*2} \mid X_{*1} = x_{*1}) \\ & = f(x_{*2} \mid x_{*1}) = f(x_{*2}, x_{*1}) / f(x_{*1}) \\ & = \frac{\alpha(\alpha+1)}{(x_{*1} + x_{*2} - 1)^{\alpha+2}} \Big/ \frac{\alpha}{x_{*1}^{\alpha+1}} = \frac{(\alpha+1)x_{*1}^{\alpha+1}}{(x_{*1} + x_{*2} - 1)^{\alpha+2}} \quad \text{(S1.10)} \\ & (x_{*1} \geq 1,\ x_{*2} \geq 1,\ \alpha > 0), \end{aligned}$$

which shows that the distribution of $x_{*1} + X_{*2} - 1$ is $P_{\mathrm{I}}(x_{*1},\ \alpha + 1)$. Then,

$$\begin{aligned} & \mathrm{E}\{X_{*1}(X_{*1} + X_{*2} - 1)\} \\ & = \mathrm{E}_{X_{*1}}[X_{*1}\{\mathrm{E}_{X_{*2}|X_{*1}}(x_{*1} + X_{*2} - 1 \mid X_{*1} = x_{*1})\}_{x_{*1} = X_{*1}}] \\ & \equiv \mathrm{E}\{X_{*1}\mathrm{E}(X_{*1} + X_{*2} - 1 \mid X_{*1})\} \\ & = \mathrm{E}\left(X_{*1}\frac{\alpha+1}{\alpha}X_{*1}\right) = \frac{\alpha+1}{\alpha} \times \frac{\alpha}{\alpha-2} = \frac{\alpha+1}{\alpha-2}. \end{aligned} \quad \text{(S1.11)}$$

On the other hand, the left-hand side of (S1.11) is

$$\mathrm{E}(X_{*1}^2) + \mathrm{E}(X_{*1}X_{*2}) - \mathrm{E}(X_{*1})$$

$$= \frac{\alpha}{\alpha-2} - \frac{\alpha}{\alpha-1} + \mathrm{E}(X_{*1}X_{*2}) = \frac{\alpha}{(\alpha-2)(\alpha-1)} + \mathrm{E}(X_{*1}X_{*2}). \quad \text{(S1.12)}$$

From (S1.11) and (S1.12),

$$\mathrm{E}(X_{*1}X_{*2}) = \frac{\alpha+1}{\alpha-2} - \frac{\alpha}{(\alpha-2)(\alpha-1)} = \frac{\alpha^2-\alpha-1}{(\alpha-2)(\alpha-1)}, \quad \text{(S1.13)}$$

which gives (S1.9). Q.E.D.

Proof 2. It is known that for non-negative random variables X_1^* and X_2^*, it holds that $\mathrm{E}(X_1^* X_2^*) = \int_0^{+\infty}\int_0^{+\infty} S(x_1^*, x_2^*)\,\mathrm{d}x_1^*\mathrm{d}x_2^*$ when the expectation exists (Hoeffding, 1940; Nadarajah & Mitov, 2003, Theorem 2; Ogasawara, 2019b, Equation (3.5)). Using this formula,

$$\mathrm{E}(X_{*1}X_{*2}) = \int_0^{+\infty}\int_0^{+\infty} S(x_{*1}, x_{*2})\,\mathrm{d}x_{*1}\mathrm{d}x_{*2}$$

$$\begin{aligned} &= \int_0^1\int_0^1 \mathrm{d}x_{*1}\mathrm{d}x_{*2} + 2\int_0^1\left\{\int_1^{+\infty}(1/x_{*1}^{\alpha})\mathrm{d}x_{*1}\right\}\mathrm{d}x_{*2} \\ &\quad + \int_1^{+\infty}\int_1^{+\infty}\{1/(x_{*1}+x_{*2}-1)^{\alpha}\}\,\mathrm{d}x_{*1}\mathrm{d}x_{*2} \end{aligned} \quad \text{(S1.14)}$$

$$= 1 + 2\left[-\frac{1}{(\alpha-1)x_{*1}^{\alpha-1}}\right]_1^{+\infty} + \left[\left[\frac{1}{(\alpha-1)(\alpha-2)(x_{*1}+x_{*2}-1)^{\alpha-2}}\right]_1^{+\infty}\right]_1^{+\infty}$$

$$= 1 + \frac{2}{\alpha-1} + \frac{1}{(\alpha-2)(\alpha-1)} = \frac{\alpha^2-3\alpha+2+2(\alpha-2)+1}{(\alpha-2)(\alpha-1)}$$

$$= \frac{\alpha^2-\alpha-1}{(\alpha-2)(\alpha-1)} \quad (\alpha > 2).$$

Note that although the regions of the above integrals are wider than the support of $X_{*1} \ge 1$ and $X_{*2} \ge 1$, the integrals are well defined. Q.E.D.

The covariance of X_{*1} and X_{*2} is given from Lemmas 1 and 2 as

$$\mathrm{cov}(X_{*1}, X_{*2}) = \frac{\alpha^2 - \alpha - 1}{(\alpha-1)(\alpha-2)} - \left(\frac{\alpha}{\alpha-1}\right)^2$$

$$= \frac{(\alpha^2 - \alpha - 1)(\alpha-1) - \alpha^2(\alpha-2)}{(\alpha-1)^2(\alpha-2)} = \frac{1}{(\alpha-1)^2(\alpha-2)}, \quad \text{(S1.15)}$$

which gives

$$\mathrm{cov}(X_1^*, X_2^*) = \frac{\beta_1\beta_2}{(\alpha-1)^2(\alpha-2)}. \quad \text{(S1.16)}$$

The variance of X_{*i} is given from Lemma 1 as

$$\mathrm{var}(X_{*i}) = \frac{\alpha}{\alpha-2} - \left(\frac{\alpha}{\alpha-1}\right)^2 = \frac{\alpha\{(\alpha-1)^2 - \alpha(\alpha-2)\}}{(\alpha-1)^2(\alpha-2)}$$

$$= \frac{\alpha}{(\alpha-1)^2(\alpha-2)} \quad \text{(S1.17)}$$

with $\mathrm{var}(X_i^*) = \dfrac{\beta_i^2\alpha}{(\alpha-1)^2(\alpha-2)}\ (i = 1,\ 2)$. From these results, the correlation coefficient of X_1^* and X_2^* is given by

$$\mathrm{cor}(X_1^*, X_2^*) = 1/\alpha < 0.5\ \ (\alpha > 2). \quad \text{(S1.18)}$$

The above variances and correlation coefficient are well-known (Mardia, 1962).

Let $\mathrm{E}(X_i^*) = \mu_i^*,\ \mathrm{E}(X_{*i}) = \mu_{*i} = \mu_i^*/\beta_i = \alpha/(\alpha-1)\ (i = 1,\ 2)$ and $\sigma_{*(m_1,m_2)} = \mathrm{E}\{(X_{*1} - \mu_{*1})^{m_1}(X_{*2} - \mu_{*2})^{m_2}\}\ (m_1 > 0,\ m_2 > 0,\ \alpha > m_1 + m_2)$.

Theorem S1.

$$\sigma_{*(m-1,\,1)} = \sigma_{*(m,\,0)}/\alpha\ \ (m > 1,\ \alpha > m). \quad \text{(S1.19)}$$

Proof. As in Proof 1 of Lemma 2,

$$\mathrm{E}\{(X_{*1} - \mu_{*1})^{m-1}(X_{*1} + X_{*2} - 1)\}$$

$$= \mathrm{E}\{(X_{*1} - \mu_{*1})^{m-1}\mathrm{E}(X_{*1} + X_{*2} - 1 \mid X_{*1})\}$$

$$= \mathrm{E}\left\{(X_{*1} - \mu_{*1})^{m-1}\frac{\alpha+1}{\alpha}X_{*1}\right\} \quad \text{(S1.20)}$$

$$= \frac{\alpha+1}{\alpha}\mathrm{E}\left\{(X_{*1} - \mu_{*1})^m + \mu_{*1}(X_{*1} - \mu_{*1})^{m-1}\right\}$$

$$= \frac{\alpha+1}{\alpha}\sigma_{*(m,\,0)} + \frac{\alpha+1}{\alpha}\times\frac{\alpha}{\alpha-1}\sigma_{*(m-1,\,0)}$$

$$= \frac{\alpha+1}{\alpha}\sigma_{*(m,\,0)} + \frac{\alpha+1}{\alpha-1}\sigma_{*(m-1,\,0)}.$$

On the other hand, the left-hand side of the first equation of (S1.20) is

$$\begin{aligned}
&\mathrm{E}\{(X_{*1}-\mu_{*1})^{m-1}(X_{*1}+X_{*2}-1)\} \\
&= \mathrm{E}\left[(X_{*1}-\mu_{*1})^{m-1}\{(X_{*1}-\mu_{*1})+(X_{*2}-\mu_{*2})+(\mu_{*1}+\mu_{*2}-1)\}\right] \\
&= \sigma_{*(m,\,0)} + \sigma_{*(m-1,\,1)} + \left(\frac{2\alpha}{\alpha-1}-1\right)\sigma_{*(m-1,\,0)} \\
&= \sigma_{*(m,\,0)} + \sigma_{*(m-1,\,1)} + \frac{\alpha+1}{\alpha-1}\sigma_{*(m-1,\,0)}.
\end{aligned} \tag{S1.21}$$

Equating (S1.20) with (S1.21),

$$\sigma_{*(m-1,\,1)} = \left(\frac{\alpha+1}{\alpha}-1\right)\sigma_{*(m,\,0)} = \frac{\sigma_{*(m,\,0)}}{\alpha} \tag{S1.22}$$

follows, which gives (S1.19). Q.E.D.

A special case of Theorem S1 is given by

$$\mathrm{cov}(X_{*1}, X_{*2}) = \mathrm{var}(X_{*1})/\alpha = \mathrm{var}(X_{*2})/\alpha\,. \tag{S1.23}$$

as found earlier with $\mathrm{cor}(X_1^*, X_2^*) = \mathrm{cor}(X_{*1}, X_{*2}) = 1/\alpha$. Define $\sigma^*_{(m_1,\,m_2)} = \mathrm{E}\{(X_1^*-\mu_1^*)^{m_1}(X_2^*-\mu_2^*)^{m_2}\}$. Then, we have $\sigma^*_{(m_1,\,m_2)} = \beta_1^{m_1}\beta_2^{m_2}\sigma_{*(m_1,\,m_2)}$.

Before using Theorem S1 for higher-order cross moments, we provide the proofs of the univariate third and fourth central moments of $X_{*i}\,(i=1,\,2)$, which are denoted by $\sigma_{*(3,\,0)}(=\sigma_{*(0,\,3)})$ and $\sigma_{*(4,\,0)}(=\sigma_{*(0,\,4)})$, respectively. These results give the skewness and non-excess kurtosis of $X_i^*\,(i=1,\,2)$, which are known, though their proofs are not well documented to the author's knowledge.

Theorem S2 (Johnson et al., 1994, Equations (20.11c) and (20.11d)). *The skewness and non-excess kurtosis common to* $X_i^*\,(i=1,\,2)$ *are*

$$\kappa_{(3)}^{(Z_i)} = \frac{2(\alpha+1)}{\alpha-3}\sqrt{\frac{\alpha-2}{\alpha}} \ (\alpha > 3) \ \textit{and} \tag{S1.24}$$

$$\kappa_{(4)}^{(Z_i)} + 3 = \frac{3(\alpha-2)(3\alpha^2+\alpha+2)}{\alpha(\alpha-3)(\alpha-4)} \ (\alpha > 4)\,(i = 1,\ 2), \tag{S1.25}$$

respectively, where $\kappa_{(j)}^{(X)}$ *is the j-th cumulant of variable X*; *and*

$Z_1 = (X_1^* - \mu_1^*)/\sqrt{\sigma_{(2,0)}^*} = (X_{*1} - \mu_{*1})/\sqrt{\sigma_{*(2,\,0)}}$ *and*

$Z_2 = (X_2^* - \mu_2^*)/\sqrt{\sigma_{(0,2)}^*} = (X_{*2} - \mu_{*2})/\sqrt{\sigma_{*(0,\,2)}}$ *with*

$\sigma_{(2,0)}^* = \mathrm{var}(X_1^*)$, $\sigma_{(0,\,2)}^* = \mathrm{var}(X_2^*)$ *and*

$\sigma_{*(2,\,0)} = \sigma_{*(0,\,2)} = \mathrm{var}(X_{*1}) = \mathrm{var}(X_{*2})$.

Proof. Firstly, we obtain the result for $\sigma_{*(3,\,0)} = \sigma_{*(0,\,3)}$ $= \mathrm{E}\{(X_{*i} - \mu_{*i})^3\}\ (i = 1,\ 2)$. Expanding $(X_{*i} - \mu_{*i})^3$,

$$\begin{aligned}
\sigma_{*(3,0)} &= \mathrm{E}(X_{*i}^3) - 3\mathrm{E}(X_{*i}^2)\mu_{*i} + 2\mu_{*i}^3 \\
&= \frac{\alpha}{\alpha-3} - \frac{3\alpha}{\alpha-2}\times\frac{\alpha}{\alpha-1} + \frac{2\alpha^3}{(\alpha-1)^3} \\
&= \frac{\alpha}{(\alpha-1)^3(\alpha-2)(\alpha-3)} \\
&\quad \times\{(\alpha-1)^3(\alpha-2) - 3\alpha(\alpha-1)^2(\alpha-3) + 2\alpha^2(\alpha-2)(\alpha-3)\} \\
&= \frac{\alpha}{(\alpha-1)^3(\alpha-2)(\alpha-3)}[\{-3-2-3(-3-2)+2(-5)\}\alpha^3 \qquad \text{(S1.26)} \\
&\quad + \{(-3)(-2)+3-3(1+6)+2\times2\times3\}\alpha^2 + \{-1+3(-2)-3(-3)\}\alpha + 2] \\
&= \frac{2\alpha(\alpha+1)}{(\alpha-1)^3(\alpha-2)(\alpha-3)} \ (\alpha > 3;\ i = 1,\ 2).
\end{aligned}$$

Then,

$$\begin{aligned}
\kappa_{(3)}^{(Z_i)} &= \sigma_{*(3,\,0)}/(\sigma_{*(2,\,0)})^{3/2} \\
&= \frac{2\alpha(\alpha+1)}{(\alpha-1)^3(\alpha-2)(\alpha-3)} \times \frac{(\alpha-1)^3(\alpha-2)^{3/2}}{\alpha^{3/2}} \qquad \text{(S1.27)}
\end{aligned}$$

$$= \frac{2(\alpha+1)}{\alpha-3}\sqrt{\frac{\alpha-2}{\alpha}} \ \ (\alpha > 3;\ i = 1, 2),$$

which is equal to (S1.24).

Secondly, $\sigma_{*(4,\,0)} = \mathrm{E}\{(X_{*i} - \mu_{*i})^4\}\ \ (i = 1,\ 2)$ is derived. As before,

$$\sigma_{*(4,0)} = \mathrm{E}(X_{*i}^4) - \mathrm{E}(X_{*i}^3)\mu_{*i} + 6\mathrm{E}(X_{*i}^2)\mu_{*i}^2 - 3\mu_{*i}^4$$

$$= \frac{\alpha}{\alpha-4} - 4\frac{\alpha}{\alpha-3}\times\frac{\alpha}{\alpha-1} + 6\frac{\alpha}{\alpha-2}\times\frac{\alpha^2}{(\alpha-1)^2} - 3\frac{\alpha^4}{(\alpha-1)^4}$$

$$= \frac{\alpha}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)} \qquad \text{(S1.28)}$$

$$\times\{(\alpha-1)^4(\alpha-2)(\alpha-3) - 4\alpha(\alpha-1)^3(\alpha-2)(\alpha-4)$$

$$+ 6\alpha^2(\alpha-1)^2(\alpha-3)(\alpha-4) - 3\alpha^3(\alpha-2)(\alpha-3)(\alpha-4)\}$$

$$= \frac{\alpha}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)}\{\,(\alpha^4 - 4\alpha^3 + 6\alpha^2 - 4\alpha + 1)(\alpha^2 - 5\alpha + 6)$$

$$- 4\alpha(\alpha^3 - 3\alpha^2 + 3\alpha - 1)(\alpha^2 - 6\alpha + 8) + 6\alpha^2(\alpha^2 - 2\alpha + 1)(\alpha^2 - 7\alpha + 12)$$

$$- 3\alpha^3(\alpha^3 - 9\alpha^2 + 26\alpha - 24)\}$$

$$= \frac{\alpha}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)}[\ \ (-4-5)\alpha^5 + (6+20+6)\alpha^4$$

$$+ (-24-30-4)\alpha^3 + (1+20+36)\alpha^2 + (-5-24)\alpha + 6$$

$$- 4\alpha\{(-3-6)\alpha^4 + (8+18+3)\alpha^3 + (-24-18-1)\alpha^2 + (6+24)\alpha - 8\}$$

$$+ 6\alpha^2\{(-2-7)\alpha^3 + (1+14+12)\alpha^2 + (-7-24)\alpha + 12\}$$

$$+ 27\alpha^5 - 78\alpha^4 + 72\alpha^3\ \ ]$$

$$= \frac{\alpha}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)}\{\,(-9+36-54+27)\alpha^5$$

$$+ (32-116+162-78)\alpha^4 + (-58+172-186+72)\alpha^3$$

$$+ (57-120+72)\alpha^2 + (-29+32)\alpha + 6\,\}$$

$$= \frac{\alpha(9a^2 + 3\alpha + 6)}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)} = \frac{3\alpha(3a^2 + \alpha + 2)}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)}.$$

Consequently,

$$\kappa_{(4)}^{(Z_i)}+3=\frac{3\alpha(3\alpha^2+\alpha+2)}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)}\times\frac{(\alpha-1)^4(\alpha-2)^2}{\alpha^2} \qquad \text{(S1.29)}$$
$$=\frac{3(\alpha-2)(3\alpha^2+\alpha+2)}{\alpha(\alpha-3)(\alpha-4)} \quad (\alpha>4;\ i=1,2)$$

follows, which is equal to (S1.25). Q.E.D.

The excess kurtosis is given by

$$\kappa_{(4)}^{(Z_i)}=\frac{3(\alpha-2)(3\alpha^2+\alpha+2)}{\alpha(\alpha-3)(\alpha-4)}-3$$
$$=\frac{3\{3\alpha^3-5\alpha^2-4-(\alpha^3-7\alpha^2+12\alpha)\}}{\alpha(\alpha-3)(\alpha-4)} \qquad \text{(S1.30)}$$
$$=\frac{3(2\alpha^3+2\alpha^2-12\alpha-4)}{\alpha(\alpha-3)(\alpha-4)}=\frac{6(\alpha^3+\alpha^2-6\alpha-2)}{\alpha(\alpha-3)(\alpha-4)}>0$$
$$(\alpha>4;\ i=1,2).$$

The positive property of (S1.30), when $\alpha>4$, is shown by the positiveness of the numerator and its first derivative when $\alpha>4$ in the last result of (S1.30).

S.2 Cross central moments of the third and fourth orders

Let $\sigma_{*11...122...2}$ (m_1 and m_2 times of 1 and 2, respectively) $=\sigma_{*(m_1,\,m_2)}$ and $\rho_{11...122...2}=\sigma_{*11...122...2}/(\sigma_{*ii})^{(m_1+m_2)/2}$ $=\sigma_{*(m_1,\,m_2)}/(\sigma_{*ii})^{(m_1+m_2)/2}$ $(i=1,\,2)$. When $m_1=m_2=1,$ ρ_{12} is the correlation coefficient of X_1^* and X_2^*. An alternative similar notation for the moments of unstandardized X_1^* and X_2^* are defined as $\sigma_{11...122...2}^*=\sigma_{(m_1,\,m_2)}^*$.

Note that in the standardized moments, the subscripts 1 and 2 can be exchanged whereas in the unstandardized moments, they cannot be exchanged unless scale parameters are the same.

Corollary S1.

$$0<\sigma_{*112}=\sigma_{*122}=\frac{2(\alpha+1)}{(\alpha-1)^3(\alpha-2)(\alpha-3)}, \qquad \text{(S2.1a)}$$

$$0 < \rho_{112} = \rho_{122} = \frac{2(\alpha+1)\sqrt{\alpha-2}}{(\alpha-3)\alpha^{3/2}} < \frac{\rho_{111}}{3} = \frac{\rho_{222}}{3} (\alpha > 3), \qquad \text{(S2.1b)}$$

$$0 < \sigma_{*1112} = \sigma_{*1222} = \frac{3(3a^2+\alpha+2)}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)}, \qquad \text{(S2.2a)}$$

$$0 < \rho_{1112} = \rho_{1222} = \frac{3(\alpha-2)(3\alpha^2+\alpha+2)}{\alpha^2(\alpha-3)(\alpha-4)} < \frac{\rho_{1111}}{4} = \frac{\rho_{2222}}{4}$$
$$(\alpha > 4). \qquad \text{(S2.2b)}$$

Proof. Equations (S2.1a) and (S2.2a) are given by Theorem S1, (S1.26) and (S1.29). Equation (S2.1b) is given by

$$\begin{aligned}\rho_{112} = \rho_{122} &= \sigma_{*122} / \sigma_{*ii}^{3/2} \\ &= \frac{2(\alpha+1)}{(\alpha-1)^3(\alpha-2)(\alpha-3)} \times \frac{(\alpha-1)^3(\alpha-2)^{3/2}}{\alpha^{3/2}} \\ &= \frac{2(\alpha+1)\sqrt{\alpha-2}}{(\alpha-3)\alpha^{3/2}} = \frac{\rho_{iii}}{\alpha} < \frac{\rho_{iii}}{3} \quad (\alpha > 3;\ i = 1,\ 2).\end{aligned} \qquad \text{(S2.3)}$$

$$\begin{aligned}\rho_{1112} = \rho_{1222} &= \sigma_{*1112} / \sigma_{*ii}^{2} \\ &= \frac{3(3a^2+\alpha+2)}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)} \times \frac{(\alpha-1)^4(\alpha-2)^2}{\alpha^2} \\ &= \frac{3(\alpha-2)(3\alpha^2+\alpha+2)}{\alpha^2(\alpha-3)(\alpha-4)} = \frac{\rho_{iiii}}{\alpha} < \frac{\rho_{iiii}}{4} \quad (\alpha > 4;\ i = 1,\ 2).\end{aligned} \qquad \text{(S2.4)}$$

The cross bivariate fourth cumulant corresponding to (S2.4) is

$$\begin{aligned}\kappa_{1112}^{(Z)} &= \rho_{1112} - 3\rho_{12} = \rho_{1222} - 3\rho_{12} \\ &= \frac{1}{\alpha^2(\alpha-3)(\alpha-4)}\{3(3\alpha^3-5\alpha^2-4) - 3\alpha(\alpha^2-7\alpha+12)\} \\ &= \frac{3(2\alpha^3+2\alpha^2-12\alpha-4)}{\alpha^2(\alpha-3)(\alpha-4)} = \frac{6(\alpha^3+\alpha^2-6\alpha-2)}{\alpha^2(\alpha-3)(\alpha-4)} > 0 \quad (\alpha > 4),\end{aligned} \qquad \text{(S2.5)}$$

where the positiveness of the final result is given as before.

Theorem S3.

$$\sigma_{*1122} = \frac{3a^3 - a^2 + 14\alpha + 4}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)} \quad \text{and} \tag{S2.6a}$$

$$\rho_{1122} = \frac{(\alpha-2)(\alpha^3 - \alpha^2 + 14\alpha + 4)}{\alpha^2(\alpha-3)(\alpha-4)} \quad (\alpha > 4). \tag{S2.6b}$$

Proof. As before

$$\begin{aligned}
&\mathrm{E}\{(X_{*1} - \mu_{*1})^2 (X_{*1} + X_{*2} - 1)^2\} \\
&= \mathrm{E}[(X_{*1} - \mu_{*1})^2 \mathrm{E}\{(X_{*1} + X_{*2} - 1)^2 \mid X_{*1})\}] \\
&= \mathrm{E}\left\{(X_{*1} - \mu_{*1})^2 \frac{\alpha+1}{\alpha-1} X_{*1}^2\right\} \\
&= \frac{\alpha+1}{\alpha-1}\mathrm{E}[\ (X_{*1} - \mu_{*1})^2 \{(X_{*1} - \mu_{*1})^2\} + 2\mu_{*1}(X_{*1} - \mu_{*1}) + \mu_{*1}^2\}\ ] \\
&= \frac{\alpha+1}{\alpha-1}\{\sigma_{*1111} + 2\mu_{*1}\sigma_{*111} + \mu_{*1}^2 \,\mathrm{var}(X_{*1})\}.
\end{aligned} \tag{S2.7}$$

On the other-hand, the left-hand side of the first equation of (S2.7) is

$$\begin{aligned}
&\mathrm{E}[\ (X_{*1} - \mu_{*1})^2 \{(X_{*2} - \mu_{*2}) + (X_{*1} - \mu_{*1}) + \mu_{*1} + \mu_{*2} - 1\}^2\ ] \\
&= \sigma_{*1122} + 2\sigma_{*1112} + \sigma_{*1111} + 2(\mu_{*1} + \mu_{*2} - 1)(\sigma_{*112} + \sigma_{*111}) \\
&+ (\mu_{*1} + \mu_{*2} - 1)^2 \,\mathrm{var}(X_{*1}).
\end{aligned} \tag{S2.8}$$

Equating (S2.7) with (S2.8),

$$\begin{aligned}
\sigma_{*1122} = &-2\sigma_{*1112} + \left(\frac{\alpha+1}{\alpha-1} - 1\right)\sigma_{*1111} - 2(\mu_{*1} + \mu_{*2} - 1)\sigma_{*112} \\
&+ 2\left\{\frac{\alpha+1}{\alpha-1}\mu_{*1} - (\mu_{*1} + \mu_{*2} - 1)\right\}\sigma_{*111} \\
&+ \left\{\frac{\alpha+1}{\alpha-1}\mu_{*1}^2 - (\mu_{*1} + \mu_{*2} - 1)^2\right\}\mathrm{var}(X_{*1})
\end{aligned} \tag{S2.9}$$

follows.

Using $\mu_{*1} + \mu_{*2} - 1 = (\alpha+1)/(\alpha-1)$,

$$
\begin{aligned}
\sigma_{*1122} &= -2\sigma_{*1112} + \frac{2\sigma_{*1111}}{\alpha-1} - 2\frac{\alpha+1}{\alpha-1}\sigma_{*112} \\
&\quad + 2\left(\frac{\alpha+1}{\alpha-1}\times\frac{\alpha}{\alpha-1} - \frac{\alpha+1}{\alpha-1}\right)\sigma_{*111} \\
&\quad + \left\{\frac{\alpha+1}{\alpha-1}\times\frac{\alpha^2}{(\alpha-1)^2} - \frac{(\alpha+1)^2}{(\alpha-1)^2}\right\}\frac{\alpha}{(\alpha-1)^2(\alpha-2)} \\
&= -\frac{2\times 3(3\alpha^2+\alpha+2)}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)} \\
&\quad + \frac{2}{\alpha-1}\times\frac{3\alpha(3\alpha^2+\alpha+2)}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)} \\
&\quad - 2\frac{\alpha+1}{\alpha-1}\times\frac{2(\alpha+1)}{(\alpha-1)^3(\alpha-2)(\alpha-3)} \\
&\quad + 2\frac{\alpha+1}{\alpha-1}\left(\frac{\alpha}{\alpha-1} - 1\right)\frac{2\alpha(\alpha+1)}{(\alpha-1)^3(\alpha-2)(\alpha-3)} \\
&\quad + \frac{\alpha(\alpha+1)}{(\alpha-1)^4(\alpha-2)}\left\{\frac{\alpha^2}{\alpha-1} - (\alpha+1)\right\} \\
&= \frac{1}{(\alpha-1)^5(\alpha-2)(\alpha-3)(\alpha-4)}\{-6(3\alpha^2+\alpha+2)(\alpha-1) \\
&\quad + 18\alpha^3 + 6\alpha^2 + 12\alpha - 4(\alpha+1)^2(\alpha-1)(\alpha-4) + 4\alpha(\alpha+1)^2(\alpha-4) \\
&\quad + \alpha(\alpha+1)(\alpha-3)(\alpha-4)\} \\
&= \frac{1}{(\alpha-1)^5(\alpha-2)(\alpha-3)(\alpha-4)}[-18\alpha^3 + 12\alpha^2 - 6\alpha + 12 \\
&\quad + 18\alpha^3 + 6\alpha^2 + 12\alpha + \{-4(\alpha^2+2\alpha+1)(\alpha-1) + 4(\alpha^3+2\alpha^2+\alpha) \\
&\qquad + \alpha^3 - 2\alpha^2 - 3\alpha\}(\alpha-4)\,]
\end{aligned}
\tag{S2.10}
$$

$$= \frac{1}{(\alpha-1)^5(\alpha-2)(\alpha-3)(\alpha-4)}$$
$$\times\{18\alpha^2+6\alpha+12+(\alpha^3+2\alpha^2+5\alpha+4)(\alpha-4)\}$$
$$= \frac{1}{(\alpha-1)^5(\alpha-2)(\alpha-3)(\alpha-4)}$$
$$\times\{18\alpha^2+6\alpha+12+(\alpha^4-2\alpha^3-3\alpha^2-16\alpha-16)\}$$
$$= \frac{\alpha^4-2\alpha^3+15\alpha^2-10\alpha-4}{(\alpha-1)^5(\alpha-2)(\alpha-3)(\alpha-4)} = \frac{(\alpha-1)(\alpha^3-\alpha^2+14\alpha+4)}{(\alpha-1)^5(\alpha-2)(\alpha-3)(\alpha-4)}$$
$$= \frac{\alpha^3-\alpha^2+14\alpha+4}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)} \quad (\alpha>4;\ i=1,\ 2),$$

which gives (S2.6a). From this result,

$$\begin{aligned}\rho_{1122} &= \sigma_{*1122}/\sigma_{*ii}^2 \\ &= \frac{\alpha^3-\alpha^2+14\alpha+4}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)} \times \frac{(\alpha-1)^4(\alpha-2)^2}{\alpha^2} \\ &= \frac{(\alpha-2)(\alpha^3-\alpha^2+14\alpha+4)}{\alpha^2(\alpha-3)(\alpha-4)} = \ (\alpha>4;\ i=1,\ 2),\end{aligned} \tag{S2.11}$$

giving (S2.6b). Q.E.D.

Theorem S3 gives

$$\begin{aligned}\kappa_{1122}^{(Z)} &= \rho_{1122}-1-2\rho_{12}^2 \\ &= \frac{(\alpha-2)(\alpha^3-\alpha^2+14\alpha+4)}{\alpha^2(\alpha-3)(\alpha-4)} - 1 - \frac{2}{\alpha^2} \\ &= \frac{1}{\alpha^2(\alpha-3)(\alpha-4)}\{\alpha^4-3\alpha^3+16\alpha^2-24\alpha-8 \\ &\quad +(-\alpha^4+7\alpha^3-12\alpha^2)+(-2\alpha^2+14\alpha-24)\} \\ &= \frac{4\alpha^3+2\alpha^2-10\alpha-32}{\alpha^2(\alpha-3)(\alpha-4)} > 0 \ \ (\alpha>4).\end{aligned} \tag{S2.12}$$

S.3 General formulas for the higher-order cross moments

First, the following basic result is given.

Lemma S3. *For a positive integer m with* $\alpha > m$,

$$\sigma_{*(m,\,0)} = \sigma_{*(0,\,m)} = \mathrm{E}\{(X_{*i} - \mu_{*i})^m\}$$

$$= \sum_{j=0}^{m} {}_mC_j \mathrm{E}(X_{*i}^j)(-\mu_{*i}^j)^{m-j} = \sum_{j=0}^{m} {}_mC_j \frac{\alpha}{\alpha - j}\left(-\frac{\alpha}{\alpha-1}\right)^{m-j} \quad \text{(S3.1)}$$

$$= \sum_{j=0}^{m} {}_mC_j (-1)^{m-j} \frac{\alpha^{m-j+1}}{(\alpha - j)(\alpha-1)^{m-j}}.$$

Let $\rho_{(i)}$ be the *i*-th order univariate central moment for a standardized variable with unit variance:

$$\rho_{(i)} = \sigma_{*(i,0)} / (\sigma_{*jj})^{i/2} = \sigma_{*(0,i)} / (\sigma_{*jj})^{i/2} = \sigma_{*(i,0)} / \{\mathrm{var}(X_{*j})\}^{i/2}, \quad \text{(S3.2)}$$

where $\mathrm{var}(X_{*j}) = \alpha / \{(\alpha-1)^2(\alpha-2)\}$ and

$$\rho_{(2)} = \rho_{ii} = \sigma_{*(2,0)} / \sigma_{*jj} = \sigma_{*(0,2)} / \sigma_{*jj} = \sigma_{*ii} / \sigma_{*jj} = 1,$$
$$\rho_{(3)} = \rho_{iii}, \;\; \rho_{(4)} = \rho_{iiii} \;\; (i, j = 1, 2), \quad \text{(S3.3)}$$

as shown earlier.

Theorem S4. *For a positive integer m with* $\alpha > m$,

$$\sigma_{*(m-i,\,i)} = \sigma_{*(i,\,m-i)} = \mathrm{E}\{(X_{*1} - \mu_{*1})^{m-i}(X_{*2} - \mu_{*2})^i\}$$

$$= \frac{\alpha+1}{\alpha+1-i} \sum_{j=0}^{i} {}_iC_j \sigma_{*(m-i+j,\,0)} \mu_{*1}^{i-j}$$

$$- i! \sum_{\substack{j_1=0 \\ 0 \le j_1 + j_2 \le i}}^{i-1} \sum_{j_2=0}^{i} \frac{(\mu_{*1} + \mu_{*2} - 1)^{i-j_1-j_2} \sigma_{*(m-i+j_2,\,j_1)}}{j_1!\,j_2!\,(i-j_1-j_2)!} \quad \text{(S3.4)}$$

$$(i = 1, \ldots, [(m+1)/2]),$$

where $[\cdot]$ *is the floor function; and* $\sigma_{*(m-j,\,j)}$ $(j = 1, \ldots, i-1)$ *and the raw univariate moments up to the m-th order are assumed to be given*.

Proof. As before

$$
\begin{aligned}
&\mathrm{E}\{(X_{*1}-\mu_{*1})^{m-i}(X_{*1}+X_{*2}-1)^{i}\}\\
&=\mathrm{E}[(X_{*1}-\mu_{*1})^{m-i}\,\mathrm{E}\{(X_{*1}+X_{*2}-1)^{i}\mid X_{*1})\}]\\
&=\mathrm{E}\left\{(X_{*1}-\mu_{*1})^{m-i}\,\frac{\alpha+1}{\alpha+1-i}X_{*1}^{i}\right\}\\
&=\frac{\alpha+1}{\alpha+1-i}\mathrm{E}\left[(X_{*1}-\mu_{*1})^{m-i}\sum\nolimits_{j=0}^{i}{}_{i}C_{j}(X_{*1}-\mu_{*1})^{j}\,\mu_{*1}^{i-j}\right]\\
&=\frac{\alpha+1}{\alpha+1-i}\sum\nolimits_{j=0}^{i}{}_{i}C_{j}\{\mathrm{E}(X_{*1}-\mu_{*1})^{m-i+j}\}\mu_{*1}^{i-j}\\
&=\frac{\alpha+1}{\alpha+1-i}\sum\nolimits_{j=0}^{i}{}_{i}C_{j}\sigma_{*(m-i+j,\,0)}\mu_{*1}^{i-j}\quad(i=1,\ldots,[(m+1)/2]).
\end{aligned}
\tag{S3.5}
$$

On the other hand, the left-hand side of (3.5) is

$$
\begin{aligned}
&\mathrm{E}\{(X_{*1}-\mu_{*1})^{m-i}(X_{*1}+X_{*2}-1)^{i}\}\\
&=\mathrm{E}[\,(X_{*1}-\mu_{*1})^{m-i}\{(X_{*2}-\mu_{*2})+(X_{*1}-\mu_{*1})+\mu_{*1}+\mu_{*2}-1\}^{i}\,]\\
&=\mathrm{E}\Bigg\{\;(X_{*1}-\mu_{*1})^{m-i}\sum_{\substack{j_1=0\\0\le j_1+j_2\le i}}^{i}\sum_{j_2=0}^{i}\frac{i!}{j_1!\,j_2!(i-j_1-j_2)!}(X_{*2}-\mu_{*2})^{j_1}\\
&\qquad\times(X_{*1}-\mu_{*1})^{j_2}(\mu_{*1}+\mu_{*2}-1)^{i-j_1-j_2}\;\Bigg\}
\end{aligned}
\tag{S3.6}
$$

$$
\begin{aligned}
&=\mathrm{E}\Bigg\{\;(X_{*1}-\mu_{*1})^{m-i}(X_{*2}-\mu_{*2})^{i}\\
&\qquad+\sum_{\substack{j_1=0\\0\le j_1+j_2\le i}}^{i-1}\sum_{j_2=0}^{i}\frac{i!}{j_1!\,j_2!(i-j_1-j_2)!}(X_{*2}-\mu_{*2})^{j_1}\\
&\qquad\times(X_{*1}-\mu_{*1})^{m-i+j_2}(\mu_{*1}+\mu_{*2}-1)^{i-j_1-j_2}\;\Bigg\}\\
&=\sigma_{*(m-i,\,i)}+i!\sum_{\substack{j_1=0\\0\le j_1+j_2\le i}}^{i-1}\sum_{j_2=0}^{i}\frac{(\mu_{*1}+\mu_{*2}-1)^{i-j_1-j_2}}{j_1!\,j_2!(i-j_1-j_2)!}\sigma_{*(m-i+j_2,\,j_1)}.
\end{aligned}
$$

Equating (S3.5) with (S3.6), (S3.4) follows. Q.E.D.

From (S3.4), $\sigma^*_{(m-i,\,i)} = \beta_1^{m-i}\beta_2^i \sigma_{*(m-i,\,i)}$ $(i=1,...,[(m+1)/2])$. We also have

$$\rho_{(m-i,\,i)} = \frac{\sigma_{*(m-i,\,i)}}{\{\mathrm{var}(X_{*j})\}^{m/2}} = \frac{(\alpha-1)^m(\alpha-2)^{m/2}}{\alpha^{m/2}}\sigma_{*(m-i,\,i)}$$
$$(i=1,...,[(m+1)/2];\ j=1,\ 2). \tag{S3.7}$$

The raw cross moments of the *m*-order are given as follows.

Corollary S2. *For a positive integer m with* $\alpha > m$,

$$\mu_{*(m-i,\,i)} = \mu_{*(i,\,m-i)} = \mathrm{E}(X_{*1}^{m-i}X_{*2}^{i})$$
$$= \frac{\alpha+1}{\alpha+1-i}\mu_{*(m,\,0)} - i!\sum_{\substack{j_1=0 \\ 0\le j_1+j_2\le i}}^{i-1}\sum_{j_2=0}^{i}\frac{(-1)^{i-j_1-j_2}\mu_{*(m-i+j_2,\,j_1)}}{j_1!\,j_2!(i-j_1-j_2)!} \tag{S3.8}$$
$$(i=1,...,[(m+1)/2]),$$

where the cross raw moments up to the $(m-1)$*-th order are assumed to be given*.

Proof. This can be seen as a special case of Theorem 4, where $\mu_{*1}=\mu_{*2}=0$. When the definition $0^0 = 1$ is used in Theorem S4, (S3.8) follows. Q.E.D.

We note that $\mu^*_{(m-i,\,i)} = \mathrm{E}(X_1^{*(m-i)}X_2^{*i}) = \beta_1^{m-i}\beta_2^i\mu_{*(m-i,\,i)}$ $= \beta_1^{m-i}\beta_2^i\mu_{*(i,\,m-i)}$ $(i=1,...,[(m+1)/2])$.

As in Proof 2 of Lemma S2, it is known that

$$\mathrm{E}(X_{*2}^{m_1}X_{*1}^{m_2}) = \mathrm{E}(X_{*1}^{m_1}X_{*2}^{m_2})$$
$$= m_1m_2\int_0^{+\infty}\int_0^{+\infty} x_{*1}^{m_1-1}x_{*2}^{m_2-1}S(x_{*1},x_{*2})\mathrm{d}x_{*1}\mathrm{d}x_{*2}\ (m_1>0,\ m_2>0) \tag{S3.9}$$

(see the references after (S1.13)). Using $y_i \equiv x_{*i}\ (i=1,\ 2)$ and recalling that $S(y_i)=1/y_i^{\alpha}$ and $S(y_1,\ y_2)=1/(y_1+y_2-1)^{\alpha}$ $(y_i\ge 1;\ i=1,\ 2)$, (S3.9) gives

$$\mathrm{E}(X_{*1}^{m_1}X_{*2}^{m_2}) = m_1m_2\int_0^{+\infty}\int_0^{+\infty} y_1^{m_1-1}y_2^{m_2-1}S(y_1,y_2)\,\mathrm{d}y_1\mathrm{d}y_2$$
$$= m_1m_2\left\{\int_0^1\int_0^1 y_1^{m_1-1}y_2^{m_2-1}\,\mathrm{d}y_1\mathrm{d}y_2 + \int_1^{+\infty} y_1^{m_1-\alpha-1}\mathrm{d}y_1\int_0^1 y_2^{m_2-1}\mathrm{d}y_2\right. \tag{S3.10}$$

$$+\int_0^1 y_1^{m_1-1}\mathrm{d}y_1\int_1^{+\infty} y_2^{m_2-\alpha-1}\mathrm{d}y_2+\int_1^{+\infty}\int_1^{+\infty}\frac{y_1^{m_1-1}y_2^{m_2-1}}{(y_1+y_2-1)^{\alpha}}\mathrm{d}y_1\mathrm{d}y_2 \quad \Bigg\}$$

$$=1+\frac{m_1}{\alpha-m_1}+\frac{m_2}{\alpha-m_2}+m_1m_2\int_1^{+\infty}\int_1^{+\infty}\frac{y_1^{m_1-1}y_2^{m_2-1}}{(y_1+y_2-1)^{\alpha}}\mathrm{d}y_1\mathrm{d}y_2$$

$$(\alpha>m_1+m_2),$$

where the $1/(m_1m_2)$ times the last term of the last result is

$$\int_1^{+\infty}\int_1^{+\infty}\frac{y_1^{m_1-1}y_2^{m_2-1}}{(y_1+y_2-1)^{\alpha}}\mathrm{d}y_1\mathrm{d}y_2$$

$$=\frac{1}{(\alpha-1)(\alpha-2)}\left[\left[\frac{y_1^{m_1-1}y_2^{m_2-1}}{(y_1+y_2-1)^{\alpha-2}}\right]_1^{+\infty}\right]_1^{+\infty}$$

$$-\left(-\frac{1}{\alpha-1}\right)\int_1^{+\infty}\int_1^{+\infty}\frac{(m_1-1)y_1^{m_1-2}y_2^{m_2-1}+(m_2-1)y_1^{m_2-1}y_2^{m_2-2}}{(y_1+y_2-1)^{\alpha-1}}\mathrm{d}y_1\mathrm{d}y_2 \quad \text{(S3.11)}$$

$$-\frac{1}{(\alpha-1)(\alpha-2)}\int_1^{+\infty}\int_1^{+\infty}\frac{(m_1-1)(m_2-1)y_1^{m_1-2}y_2^{m_2-2}}{(y_1+y_2-1)^{\alpha}}\mathrm{d}y_1\mathrm{d}y_2.$$

$$=\frac{1}{(\alpha-1)(\alpha-2)}+\frac{m_1-1}{\alpha-1}\int_1^{+\infty}\int_1^{+\infty}\frac{y_1^{m_1-2}y_2^{m_2-1}}{(y_1+y_2-1)^{\alpha-1}}\mathrm{d}y_1\mathrm{d}y_2$$

$$+\frac{m_2-1}{\alpha-1}\int_1^{+\infty}\int_1^{+\infty}\frac{y_1^{m_1-1}y_2^{m_2-2}}{(y_1+y_2-1)^{\alpha-1}}\mathrm{d}y_1\mathrm{d}y_2$$

$$-\frac{(m_1-1)(m_2-1)}{(\alpha-1)(\alpha-2)}\int_1^{+\infty}\int_1^{+\infty}\frac{y_1^{m_1-2}y_2^{m_2-2}}{(y_1+y_2-1)^{\alpha}}\mathrm{d}y_1\mathrm{d}y_2.$$

Note that the first expression and the last result of (S3.11) give a recursive formula. That is, the last result can be expanded further as follows, where some terms are the same.

$$
\begin{aligned}
&= \frac{1}{(\alpha-1)(\alpha-2)} \\
&+ \frac{m_1-1}{\alpha-1}\left\{ \frac{1}{(\alpha-2)(\alpha-3)} + \frac{m_1-2}{\alpha-2}\int_1^{+\infty}\int_1^{+\infty} \frac{y_1^{m_1-3} y_2^{m_2-1}}{(y_1+y_2-1)^{\alpha-2}} \mathrm{d}y_1 \mathrm{d}y_2 \right. \\
&\qquad + \frac{m_2-1}{\alpha-2}\int_1^{+\infty}\int_1^{+\infty} \frac{y_1^{m_1-2} y_2^{m_2-2}}{(y_1+y_2-1)^{\alpha-2}} \mathrm{d}y_1 \mathrm{d}y_2 \qquad \text{(S3.12)} \\
&\qquad \left. - \frac{(m_1-2)(m_2-1)}{(\alpha-2)(\alpha-3)}\int_1^{+\infty}\int_1^{+\infty} \frac{y_1^{m_1-3} y_2^{m_2-2}}{(y_1+y_2-1)^{\alpha-3}} \mathrm{d}y_1 \mathrm{d}y_2 \right\} \\
&+ \frac{m_2-1}{\alpha-1}\left\{ \frac{1}{(\alpha-2)(\alpha-3)} + \frac{m_1-1}{\alpha-2}\int_1^{+\infty}\int_1^{+\infty} \frac{y_1^{m_1-2} y_2^{m_2-2}}{(y_1+y_2-1)^{\alpha-2}} \mathrm{d}y_1 \mathrm{d}y_2 \right. \\
&\qquad + \frac{m_2-2}{\alpha-2}\int_1^{+\infty}\int_1^{+\infty} \frac{y_1^{m_1-1} y_2^{m_2-3}}{(y_1+y_2-1)^{\alpha-2}} \mathrm{d}y_1 \mathrm{d}y_2 \\
&\qquad \left. - \frac{(m_1-1)(m_2-2)}{(\alpha-2)(\alpha-3)}\int_1^{+\infty}\int_1^{+\infty} \frac{y_1^{m_1-2} y_2^{m_2-3}}{(y_1+y_2-1)^{\alpha-3}} \mathrm{d}y_1 \mathrm{d}y_2 \right\} \\
&- \frac{(m_1-1)(m_2-1)}{(\alpha-1)(\alpha-2)}\left\{ \frac{1}{(\alpha-3)(\alpha-4)} + \frac{m_1-2}{\alpha-3}\int_1^{+\infty}\int_1^{+\infty} \frac{y_1^{m_1-3} y_2^{m_2-2}}{(y_1+y_2-1)^{\alpha-3}} \mathrm{d}y_1 \mathrm{d}y_2 \right. \\
&\qquad + \frac{m_2-2}{\alpha-3}\int_1^{+\infty}\int_1^{+\infty} \frac{y_1^{m_2-2} y_2^{m_2-3}}{(y_1+y_2-1)^{\alpha-3}} \mathrm{d}y_1 \mathrm{d}y_2 \\
&\qquad \left. - \frac{(m_1-2)(m_2-2)}{(\alpha-3)(\alpha-4)}\int_1^{+\infty}\int_1^{+\infty} \frac{y_1^{m_1-3} y_2^{m_2-3}}{(y_1+y_2-1)^{\alpha-4}} \mathrm{d}y_1 \mathrm{d}y_2 \right\}.
\end{aligned}
$$

This sequence may be continued until the power terms of order greater than 0 in the numerators of the last result vanish though the result soon becomes complicated. The above formula was used earlier when $m_1 = 1$ and $m_2 = 1$ in Proof 2 of Lemma S2. The result with $m_1 = 2$ and $m_2 = 1$ is derived using (S3.11) as follows for illustration.

For this case, (S3.11) becomes

$$\int_1^{+\infty}\int_1^{+\infty}\frac{y_1}{(y_1+y_2-1)^{\alpha}}\mathrm{d}y_1\mathrm{d}y_2$$

$$=\frac{1}{(\alpha-1)(\alpha-2)}+\frac{1}{\alpha-1}\int_1^{+\infty}\int_1^{+\infty}\frac{1}{(y_1+y_2-1)^{\alpha-1}}\mathrm{d}y_1\mathrm{d}y_2 \qquad \text{(S3.13)}$$

$$=\frac{1}{(\alpha-1)(\alpha-2)}+\frac{1}{(\alpha-1)(\alpha-2)(\alpha-3)}.$$

Consequently,

$$\mathrm{E}(X_{*1}^2X_{*2})$$

$$=1+\frac{1}{\alpha-1}+\frac{2}{\alpha-2}+\frac{2}{(\alpha-1)(\alpha-2)}+\frac{2}{(\alpha-1)(\alpha-2)(\alpha-3)} \qquad \text{(S3.14)}$$

$$=\frac{1}{(\alpha-1)(\alpha-2)(\alpha-3)}\{\alpha^3-6\alpha^2+11\alpha-6+(\alpha-2)(\alpha-3)$$

$$+2(\alpha-1)(\alpha-3)+2(\alpha-3)+2\}$$

$$=\frac{1}{(\alpha-1)(\alpha-2)(\alpha-3)}\{\alpha^3-6\alpha^2+11\alpha-6+(\alpha^2-5\alpha+6)$$

$$+2(\alpha^2-4\alpha+3)+2\alpha-4\}$$

$$=\frac{\alpha^3-3\alpha^2+2}{(\alpha-1)(\alpha-2)(\alpha-3)}=\frac{(\alpha-1)(\alpha^2-2\alpha-2)}{(\alpha-1)(\alpha-2)(\alpha-3)}$$

$$=\frac{\alpha^2-2\alpha-2}{(\alpha-2)(\alpha-3)}.$$

When the conditional distribution is used,

$$\mathrm{E}\{X_{*1}^2(X_{*1}+X_{*2}-1)\}=\mathrm{E}\{X_{*1}^2\mathrm{E}(X_{*1}+X_{*2}-1\,|\,X_{*1})\}$$

$$=\mathrm{E}\left\{X_{*1}^2\left(\frac{\alpha+1}{\alpha}X_{*1}\right)\right\}=\frac{\alpha+1}{\alpha}\times\frac{\alpha}{\alpha-3}=\frac{\alpha+1}{\alpha-3}. \qquad \text{(S3.15)}$$

On the other hand, the left-hand side of the first equation of (S3.15) is

$$\mathrm{E}\{X_{*1}^2(X_{*1}+X_{*2}-1)\}=\mathrm{E}(X_{*1}^3)+\mathrm{E}(X_{*1}^2X_{*2})-\mathrm{E}(X_{*1}^2)$$

$$=\mathrm{E}(X_{*1}^2X_{*2})+\frac{\alpha}{\alpha-3}-\frac{\alpha}{\alpha-2}. \qquad \text{(S3.16)}$$

From (S3.15) and (S3.16),

$$\mathrm{E}(X_{*1}^{2}X_{*2})=\frac{1}{\alpha-3}+\frac{\alpha}{\alpha-2}=\frac{\alpha^{2}-2\alpha-2}{(\alpha-2)(\alpha-3)}, \qquad \text{(S3.17)}$$

which is equal to the last result of (S3.14).

When $m_1 = 2$ and $m_2 = 2$ for $\mu_{*(2,\,2)} = \mathrm{E}(X_{*1}^{2}X_{*2}^{2})$, the last result of (S3.11) is fully used:

$$\begin{aligned}
&\int_1^{+\infty}\int_1^{+\infty}\frac{y_1y_2}{(y_1+y_2-1)^{\alpha}}\mathrm{d}y_1\mathrm{d}y_2\\
&=\frac{1}{(\alpha-1)(\alpha-2)}+\frac{1}{\alpha-1}\int_1^{+\infty}\int_1^{+\infty}\frac{y_1+y_2}{(y_1+y_2-1)^{\alpha-1}}\mathrm{d}y_1\mathrm{d}y_2 \qquad \text{(S3.18)}\\
&-\frac{1}{(\alpha-1)(\alpha-2)}\int_1^{+\infty}\int_1^{+\infty}\frac{1}{(y_1+y_2-1)^{\alpha-2}}\mathrm{d}y_1\mathrm{d}y_2\\
&=\frac{1}{(\alpha-1)(\alpha-2)}+\frac{2}{(\alpha-1)(\alpha-2)(\alpha-3)}\\
&\quad+\frac{2}{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}-\frac{1}{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}\\
&=\frac{1}{(\alpha-1)(\alpha-2)}+\frac{2}{(\alpha-1)(\alpha-2)(\alpha-3)}\\
&\quad+\frac{1}{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)},
\end{aligned}$$

where (S3.13) is used. Plugging (S3.18) in (S3.10),

$$\begin{aligned}
\mathrm{E}(X_{*1}^{2}X_{*2}^{2})&=1+\frac{4}{\alpha-2}+\frac{4}{(\alpha-1)(\alpha-2)}+\frac{8}{(\alpha-1)(\alpha-2)(\alpha-3)}\\
&\quad+\frac{4}{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)} \qquad \text{(S3.19)}
\end{aligned}$$

follows.

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