Supplement to the paper “Asymptotic properties of the Bayes modal estimators of item parameters in item response theory”

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This note is to supplement Ogasawara (2013), and gives asymptotic expansions for the reciprocals of the estimated asymptotic standard errors used for studentized estimators and a simplification of Theorem 6.

(a) $t$

Define $\hat{i}_{W}^{\alpha\alpha} = (\hat{I}_{W}^{-1})_{(\alpha\alpha)}$. Then, for $t = N^{1/2}(\hat{\alpha}_{W} - \alpha_{0}) / (\hat{i}_{W}^{\alpha\alpha})^{1/2}$,

\[
(\hat{i}_{W}^{\alpha\alpha})^{-1/2} = (i_{0}^{\alpha\alpha})^{-1/2} + \frac{1}{2} (i_{0}^{\alpha\alpha})^{-3/2} \sum_{a \geq b} \frac{2 - \delta_{ab}}{2} \{I_{0}^{-1}(E_{ab} + E_{ba})I_{0}^{-1}\}_{(\alpha\alpha)} \frac{\partial i_{0ab}}{\partial \alpha_{0}}, (\hat{\alpha}_{W} - \alpha_{0})
\]

\[
+ \frac{3}{8} (i_{0}^{\alpha\alpha})^{-5/2} \left[ \sum_{a \geq b} \frac{2 - \delta_{ab}}{2} \{I_{0}^{-1}(E_{ab} + E_{ba})I_{0}^{-1}\}_{(\alpha\alpha)} \frac{\partial i_{0ab}}{\partial \alpha_{0}}, (\hat{\alpha}_{W} - \alpha_{0}) \right]^{2}
\]

\[-\frac{1}{4} (i_{0}^{\alpha\alpha})^{-3/2} \sum_{a \geq b, c > d} \frac{1}{4} (2 - \delta_{ab})(2 - \delta_{cd})
\]

\[
\times 2\{I_{0}^{-1}(E_{ab} + E_{ba})I_{0}^{-1}(E_{cd} + E_{dc})I_{0}^{-1}\}_{(\alpha\alpha)} \frac{\partial i_{0ab}}{\partial \alpha_{0}}, (\hat{\alpha}_{W} - \alpha_{0}) \frac{\partial i_{0cd}}{\partial \alpha_{0}}, (\hat{\alpha}_{W} - \alpha_{0})
\]

\[
+ \frac{1}{4} (i_{0}^{\alpha\alpha})^{-3/2} \sum_{a \geq b} \frac{2 - \delta_{ab}}{2} \{I_{0}^{-1}(E_{ab} + E_{ba})I_{0}^{-1}\}_{(\alpha\alpha)} \frac{\partial^{2} i_{0ab}}{\partial \alpha_{0}^{2}}, (\hat{\alpha}_{W} - \alpha_{0})^{<2>}
\]

\[+ O_{p}(N^{-3/2})
\]

\[
\equiv (i_{0}^{\alpha\alpha})^{-1/2} + i_{0}^{(1)}'(\hat{\alpha}_{W} - \alpha_{0}) + i_{0}^{(2)}'(\hat{\alpha}_{W} - \alpha_{0})^{<2>} + O_{p}(N^{-3/2})
\]

\[
= (i_{0}^{\alpha\alpha})^{-1/2} + i_{0}^{(1)}'(N^{-1}\eta_{0} + N^{-1/2}\Lambda^{(1)}m^{(1)} + N^{-1}\Lambda^{(2)}m^{(2)})
\]

\[+ i_{0}^{(2)}'(N^{-1/2}\Lambda^{(1)}m^{(1)})^{<2>} + O_{p}(N^{-3/2})
\]

(S.1)
\[
\begin{align*}
&= (i_{0}^{\alpha\alpha})^{-1/2} + N^{-1/2}i_{0}^{(1)} \Lambda (1) m^{(1)} + N^{-1}i_{0}^{(1)} \eta_{0} \\
&\quad + N^{-1}\{i_{0}^{(1)} \Lambda (2) m^{(2)} + i_{0}^{(2)} (\Lambda (1) m^{(1)})^{<2>}\} + O_{p}(N^{-3/2}) \\
&\equiv (i_{0}^{\alpha\alpha})^{-1/2} + N^{-1} \eta_{t0} + \sum_{j=1}^{2} N^{-j/2} \lambda^{(j)} \cdot m^{(j)} + O_{p}(N^{-3/2}),
\end{align*}
\]

where \( \sum_{a \geq b} (\cdot) = \sum_{a=1}^{q} \sum_{b=1}^{a} (\cdot) \), \( \delta_{ab} \) is the Kronecker delta, \( E_{ab} \) is a square matrix of an appropriate size whose \((a, b)\)th element is 1 with other ones being zero, and \( \Lambda^{(j)} (j = 1, 2) \) are multivariate versions of \( \Lambda^{(j)} (j = 1, 2) \), respectively (see (8.2)).

In (S.1),
\[
\begin{align*}
i_{0}^{(1)} &= \frac{1}{2} (i_{0}^{\alpha\alpha})^{-3/2} \sum_{a \geq b} (2 - \delta_{ab}) i_{0}^{\alpha\alpha} \cdot i_{0}^{ab} \frac{\partial i_{0ab}}{\partial \alpha_{0}}, \\
i_{0}^{(2)} &= \frac{3}{8} (i_{0}^{\alpha\alpha})^{-5/2} \left[ \sum_{a \geq b} (2 - \delta_{ab}) i_{0}^{\alpha\alpha} \cdot i_{0}^{ab} \frac{\partial i_{0ab}}{\partial \alpha_{0}} \right]^{<2>} \\
&\quad - \frac{1}{8} (i_{0}^{\alpha\alpha})^{-3/2} \sum_{a \geq b} \sum_{c \geq d} (2 - \delta_{ab})(2 - \delta_{cd}) \sum_{(abcd)}^{4} i_{0}^{\alpha\alpha} \cdot i_{0}^{bcd} \cdot i_{0}^{da} \frac{\partial^{2} i_{0ab}}{\partial \alpha_{0}} \otimes \frac{\partial i_{0cd}}{\partial \alpha_{0}} \quad (S.2)
\end{align*}
\]
\[
\begin{align*}
\eta_{t0} &= i_{0}^{(1)} \cdot \eta_{0}, \quad \lambda_{t}^{(1)} = i_{0}^{(1)} \cdot \Lambda (1), \quad \lambda_{t}^{(2)} = i_{0}^{(1)} \cdot \Lambda (2) + i_{0}^{(2)} \cdot \Lambda (1)^{<2>},
\end{align*}
\]

where 
\[\sum_{(abcd)}^{4} \] is the sum of four terms considering the symmetric cases for \(a, b, c\) and \(d\), and noting
\[
I_{0} = \sum_{k=1}^{K} \frac{1}{\pi_{0k}} \frac{\partial \pi_{0k}}{\partial \alpha_{0}} \cdot \frac{\partial \pi_{0k}}{\partial \alpha_{0}}, \quad \text{with} \quad \pi_{0k} \equiv (\pi_{0})_{k} \equiv \{\pi(\alpha_{0})_{k}\} ,
\]
we have
\[
\frac{\partial i_{0ab}}{\partial \alpha_{0c}} = \sum_{k=1}^{K} \left( \frac{1}{\pi_{0k}^2} \frac{\partial \pi_{0k}}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}}{\partial \alpha_{0b}} \frac{\partial \pi_{0k}}{\partial \alpha_{0c}} + 2 \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a} \partial \alpha_{0b} \partial \alpha_{0c}} \frac{\partial \pi_{0k}}{\partial \alpha_{0b}} \right),
\]
\[
\frac{\partial^2 i_{0ab}}{\partial \alpha_{0c} \partial \alpha_{0d}} = \sum_{k=1}^{K} \left\{ 2 \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a} \partial \alpha_{0c}} \frac{\partial \pi_{0k}}{\partial \alpha_{0b}} + \frac{\partial \pi_{0k}}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}}{\partial \alpha_{0b}} \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0d}^2} \right\},
\]
\[
(a, b, c, d = 1, \ldots, q).
\]

(b) \( t_H \)

Using \( \hat{H}_W \) (see (4.2b)), define \( \hat{h}_{W}^{aa}, \hat{h}_{W}^{ab}, \hat{d}_{W}^{aa}, \hat{d}_{0}^{aa} \) in the following equations:

\[
t_H = \frac{N^{1/2}(\hat{\alpha}_W - \alpha_0)}{\{(-\hat{H}_W^{-1})_{aa}\}^{1/2}} = \frac{N^{1/2}(\hat{\alpha}_W - \alpha_0)}{(-\hat{h}_W^{aa})^{1/2}},
\]
\[
\hat{h}_W^{aa} = \left\{ \sum_{k=1}^{K} \frac{1}{\pi_k} \frac{\partial^2 \hat{\pi}_k}{\partial \hat{\alpha}_W \partial \hat{\alpha}_W} + \frac{1}{\pi_k^2} \frac{\partial \hat{\pi}_k}{\partial \hat{\alpha}_W} \right\}^{(aa)},
\]
\[
\hat{D}_W = -\hat{H}_W, \quad D_0 = -H_0 = \sum_{k=1}^{K} \pi_k \left( -\frac{1}{\pi_{0k}} \frac{\partial^2 \pi_{0k}}{\partial \alpha_0 \partial \alpha_0} + \frac{1}{\pi_{0k}^2} \frac{\partial \pi_{0k}}{\partial \alpha_0} \right).
\]

Then,

\[
(-\hat{h}_W^{aa})^{-1/2} = (\hat{d}_W^{aa})^{-1/2} = (d_0^{aa})^{-1/2}
\]
\[
+ \frac{1}{2} (d_0^{aa})^{-3/2} \sum_{a \geq b} (2 - \delta_{ab}) d_0^{aa} d_0^{ab} \frac{\partial d_{0ab}}{\partial (\pi_0', \alpha_0')} \{(p - \pi_T)', (\hat{\alpha}_W - \alpha_0)\}' \]  
\[
(S.5)
\]
\[ + \left[ \frac{3}{8} \left( d_{0}^{(a)} \right)^{-5/2} \sum_{a \geq b} (2 - \delta_{ab}) d_{0}^{ab} \frac{\partial d_{0ab}}{\partial (\pi_{T}', \alpha_{0}')}, \right]^{<2>} \]
\[ - \frac{1}{8} \left( d_{0}^{(a)} \right)^{-3/2} \sum_{a \geq b} \sum_{c \geq d^{*}} (2 - \delta_{ab})(2 - \delta_{cd^{*}}) \]
\[ \times \sum_{(abcd^{*})} d_{0}^{aa} d_{0}^{bc} d_{0}^{d*} \frac{\partial d_{0ab}}{\partial (\pi_{T}', \alpha_{0}')}, \frac{\partial d_{0cd^{*}}}{\partial (\pi_{T}', \alpha_{0}')} \]
\[ + \frac{1}{4} \left( d_{0}^{(a)} \right)^{-3/2} \sum_{a \geq b} (2 - \delta_{ab}) d_{0}^{aa} d_{0}^{ab} \frac{\partial^{2} d_{0ab}}{\partial (\pi_{T}', \alpha_{0}')}, \frac{\partial^{2} d_{0cd^{*}}}{\partial (\pi_{T}', \alpha_{0}')^{<2>}} \]
\[ \equiv \left( d_{0}^{(a)} \right)^{-1/2} + d_{0}^{(1)} \{ (p - \pi_{T}'), (\hat{\alpha}_{W} - \alpha_{0}') \}^{<2>} + O_{p} (N^{-3/2}) \]
\[ = \left( d_{0}^{(a)} \right)^{-1/2} + d_{0}^{(1)} \{ N^{-1/2} m^{(1)} , (N^{-1} \eta_{0} + N^{-1/2} \Lambda^{(1)} m^{(1)} + N^{-1} \Lambda^{(2)} m^{(2)}) \}^{<2>} + O_{p} (N^{-3/2}). \]

Decomposing \( d_{0}^{(j)} (j = 1, 2) \) using subvectors as
\[ d_{0}^{(1)} = (d_{0}^{(1A)}, d_{0}^{(1B)})^{r} \quad \text{and} \quad d_{0}^{(2)} = (d_{0}^{(2AA)}, d_{0}^{(2AB)}, d_{0}^{(2BA)}, d_{0}^{(2BB)})^{r}, \] (S.6)
(S.5) becomes
\[ = \left( d_{0}^{(a)} \right)^{-1/2} + N^{-1/2} \{ d_{0}^{(1A)} + d_{0}^{(1B)} \Lambda^{(1)} m^{(1)} + N^{-1} d_{0}^{(1B)} \eta_{0} \]
\[ + N^{-1} \{ d_{0}^{(1B)} \Lambda^{(2)} + d_{0}^{(2AA)} + 2 d_{0}^{(2AB)} (I (K) \otimes \Lambda^{(1)}) + d_{0}^{(2BB)} \Lambda^{(1)<2>} \} m^{(1)<2>} \]
\[ + O_{p} (N^{-3/2}) \] (S.7)
\[ \equiv \left( d_{0}^{(a)} \right)^{-1/2} + N^{-1} \eta_{H0} + \sum_{j=1}^{2} N^{-j/2} \lambda_{H}^{(j)} m^{(j)} + O_{p} (N^{-3/2}), \]
where
\[ \eta_{H0} = \mathrm{d}_{0}^{(1B)} \eta_{0}, \quad \lambda_{H}^{(1)} = \mathrm{d}_{0}^{(1A)} \Lambda^{(1)} + \mathrm{d}_{0}^{(1B)} \Lambda^{(1)} \]
\[ \lambda_{H}^{(2)} = \mathrm{d}_{0}^{(1B)} \Lambda^{(2)} + \mathrm{d}_{0}^{(2AA)} + 2 \mathrm{d}_{0}^{(2AB)} (I (K) \otimes \Lambda^{(1)}) + \mathrm{d}_{0}^{(2BB)} \Lambda^{(1)<2>}, \] (S.8)
and \( I (K) \) is the \( K \times K \) identity matrix.

In (S.5),
\[
\frac{\partial d_{0ab}}{\partial \pi_{Tk}} = -\frac{1}{\pi_{0k}^*} \frac{\partial^2 \pi_{0k}^*}{\partial \alpha_{0a} \partial \alpha_{0b}} + \frac{1}{\pi_{0k}^*} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}},
\]

\[
\frac{\partial d_{0ab}}{\partial \alpha_{0c}} = \sum_{k=1}^{K} \pi_{Tk} \left( -\frac{2}{\pi_{0k}^*} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}} + \frac{1}{\pi_{0k}^*} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0c}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0c}} \right) + \sum_{(ab)} \frac{1}{\pi_{0k}^*} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0c}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}} - \frac{1}{\pi_{0k}^*} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0c}} \right),
\]

\[
\frac{\partial^2 d_{0ab}}{\partial \pi_{Tk} \partial \pi_{Ti}} = 0,
\]

\[
\frac{\partial^2 d_{0ab}}{\partial \pi_{Tk} \partial \alpha_{0c}} = -\frac{2}{\pi_{0k}^*} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}} + \frac{1}{\pi_{0k}^*} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0c}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0c}} \right) + \sum_{(ab)} \frac{1}{\pi_{0k}^*} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0c}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}} - \frac{1}{\pi_{0k}^*} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0c}} \right),
\]

\[
\frac{\partial^2 d_{0ab}}{\partial \alpha_{0c} \partial \alpha_{0d}^*} = \sum_{k=1}^{K} \pi_{Tk} \left( -\frac{6}{\pi_{0k}^*} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0c}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0d}^*} + \frac{\partial \pi_{0k}^*}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0c}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0d}^*} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0d}^*} \right) \right)
\]

\[
\frac{\partial^2 d_{0ab}}{\partial \alpha_{0a} \partial \alpha_{0b} \partial \alpha_{0c} \partial \alpha_{0d}^*} = \sum_{k=1}^{K} \pi_{Tk} \left( -\frac{2}{\pi_{0k}^*} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0c}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0d}^*} + \frac{\partial \pi_{0k}^*}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0c}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0d}^*} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0d}^*} \right) + \sum_{(ab)} \frac{1}{\pi_{0k}^*} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0c}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}} - \frac{1}{\pi_{0k}^*} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0a}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0b}} \frac{\partial \pi_{0k}^*}{\partial \alpha_{0c}} \right),
\]

\[
(a, b, c, d^* = 1, ..., q; k^*, l = 1, ..., K).
\]

(c) \( t_G \)
Using $\hat{G}_w$ (see (4.2c)), define $\hat{g}^{aa}_w$, $G_0$ and $g_0^{aa}$ as

$$t_G = \frac{N^{1/2}(\hat{\alpha}_w - \alpha_0)}{((\hat{G}_w^{-1})_{aa})^{1/2}} = \frac{N^{1/2}(\hat{\alpha}_w - \alpha_0)}{(g^{aa}_w)^{1/2}},$$

$$G_0 = \sum_{k=1}^{K} \frac{\pi_{Tk}}{\pi_{0k}^2} \frac{\partial \pi_{0k}}{\partial \alpha_k} \frac{\partial \pi_{0k}}{\partial \alpha_k'}, \quad (S.10)$$

and $g_0^{aa} = (G_0^{-1})_{aa}$. Then,

$$(g^{aa}_w)^{-1/2} = (g^{aa}_0)^{-1/2}$$

$$+ \frac{1}{2} (g^{aa}_0)^{-3/2} \sum_{a \neq b} (2 - \delta_{ab}) g^{ab}_0 \frac{\partial g^{ab}_0}{\partial (\pi'_T, \alpha'_0)} \{(p - \pi'_T), (\hat{\alpha}_w - \alpha_0)\}' \{2\}$$

$$- \frac{3}{8} (g^{aa}_0)^{-5/2} \left[ \sum_{a \neq b} (2 - \delta_{ab}) g^{aa}_0 \frac{\partial g^{ab}_0}{\partial (\pi'_T, \alpha'_0)} \right]^{<2>}$$

$$+ \frac{1}{4} (g^{aa}_0)^{-3/2} \sum_{a \neq b} (2 - \delta_{ab}) g^{aa}_0 \frac{\partial^2 g^{ab}_0}{\partial (\pi'_T, \alpha'_0)} \{(p - \pi'_T), (\hat{\alpha}_w - \alpha_0)\}'^{<2>}$$

$$\equiv \bigg( g^{aa}_0 \bigg)^{-1/2} + g^{(1)}_0 \{(p - \pi'_T), (\hat{\alpha}_w - \alpha_0)\}' + g^{(2)}_0 \{(p - \pi'_T), (\hat{\alpha}_w - \alpha_0)\}'^{<2>} + O_p \left( N^{-3/2} \right)$$

$$= \bigg( g^{aa}_0 \bigg)^{-1/2} + g^{(1)}_0 \left\{ N^{-1/2} m^{(1)}, \left( N^{-1/2} \Lambda^{(1)} \right)^{m^{(1)}} + N^{-1/2} \Lambda^{(2)} m^{(2)} \right\}'$$

$$+ g^{(2)}_0 \left\{ N^{-1/2} m^{(1)}, N^{-1/2} (\Lambda^{(1)} m^{(1)}) \right\}'^{<2>} + O_p \left( N^{-3/2} \right).$$

Decomposing $g^{(j)}_0$ ($j = 1, 2$) using subvectors as

$g^{(1)}_0 = (g^{(1A)}_0, g^{(1B)}_0)'$ and $g^{(2)}_0 = (g^{(2AA)}_0, g^{(2AB)}_0, g^{(2BA)}_0, g^{(2BB)}_0)'$,

(S.11) becomes
\[
= (g_0^{(a)})^{-1/2} + N^{-1/2} (g_0^{(1A)} + g_0^{(1B)} \Lambda^{(1)}) m^{(1)} + N^{-1} g_0^{(1B)} \eta_0 \\
+ N^{-1} \{ g_0^{(1B)} \Lambda^{(2)} + g_0^{(2AA)} + 2g_0^{(2AB)} (I_{(K)} \otimes \Lambda^{(1)}) + g_0^{(2BB)} \Lambda^{(1)2<} \} m^{(1)<2>} \\
+ O_p (N^{-3/2})
\]
\[
= (g_0^{(a)})^{-1/2} + N^{-1} \eta_{G0} + \sum_{j=1}^{2} N^{-j/2} \lambda_G^{(j)} m^{(j)} + O_p (N^{-3/2}),
\]
where
\[
\eta_{G0} = g_0^{(1B)} \eta_0, \ 
\lambda_G^{(1)} = g_0^{(1A)} + g_0^{(1B)} \Lambda^{(1)},
\]
\[
\lambda_G^{(2)} = g_0^{(1B)} \Lambda^{(2)} + g_0^{(2AA)} + 2g_0^{(2AB)} (I_{(K)} \otimes \Lambda^{(1)}) + g_0^{(2BB)} \Lambda^{(1)2<}.
\]
In (S.11),
\[
\frac{\partial g_{ab}}{\partial \pi_{Tk}} = \frac{1}{\pi_{Tk}^{2}} \frac{\partial \pi_{Tk} \partial \pi_{Tk}^{*}}{\partial \alpha_{0a} \partial \alpha_{0b}},
\]
\[
\frac{\partial g_{ab}}{\partial \alpha_{0c}} = \sum_{k=1}^{K} \pi_{Tk} \left( -2 \frac{\partial \pi_{Tk} \partial \pi_{Tk}^{*}}{\partial \alpha_{0a} \partial \alpha_{0b}} \frac{\partial \pi_{Tk} \partial \alpha_{0c}}{\partial \alpha_{0b}} \right) + \frac{1}{\pi_{Tk}^{2}} \sum_{(ab)}^{2} \frac{\partial^{2} \pi_{Tk} \partial \pi_{Tk}^{*}}{\partial \alpha_{0a} \partial \alpha_{0b} \partial \alpha_{0c}},
\]
\[
\frac{\partial^{2} g_{ab}}{\partial \pi_{Tk} \partial \pi_{Tl}} = 0,
\]
\[
\frac{\partial^{2} g_{ab}}{\partial \alpha_{0c} \partial \alpha_{0d}^{*}} = \sum_{k=1}^{K} \pi_{Tk} \left( 6 \frac{\partial \pi_{Tk} \partial \pi_{Tk}^{*}}{\partial \alpha_{0a} \partial \alpha_{0b}} \frac{\partial \pi_{Tk} \partial \alpha_{0c}}{\partial \alpha_{0c}} \frac{\partial \pi_{Tk} \partial \alpha_{0d}^{*}}{\partial \alpha_{0b}} \right) - \frac{2}{\pi_{Tk}^{3}} \sum_{(abcd^{*})}^{4} \frac{\partial^{2} \pi_{Tk} \partial \pi_{Tk}^{*}}{\partial \alpha_{0a} \partial \alpha_{0c} \partial \alpha_{0b} \partial \alpha_{0d}^{*}} + \frac{1}{\pi_{Tk}^{2}} \frac{\partial^{2} \pi_{Tk}^{*}}{\partial \alpha_{0a} \partial \alpha_{0b} \partial \alpha_{0c} \partial \alpha_{0d}^{*}} \right) \right) \right)
\]
\[
(1, 2, 3, 4)^{a, b, c, d} = 1, \ldots, q; k^{*}, l = 1, \ldots, K).
\]
(d) \( t_R \)

Define \( \hat{r}_{waa} \) as

\[
\begin{align*}
\hat{t}_R & \equiv \frac{N^{1/2} (\hat{\alpha}_w - \alpha_0)}{(\partial \hat{\alpha}_w \Omega_T' \frac{\partial \hat{\alpha}_w}{\partial p})^{1/2}} \equiv \frac{N^{1/2} (\hat{\alpha}_w - \alpha_0)}{\hat{r}_{waa}^{1/2}},
\end{align*}
\]

(S.15)

where

\[
\frac{\partial \hat{\alpha}_w}{\partial p} = \frac{\partial \alpha_w(p)}{\partial p} \bigg|_{p=p}.
\]

Recalling

\[
\frac{\partial \alpha_w}{\partial \pi_T} = \frac{\partial \alpha_w(p)}{\partial p} \bigg|_{u=u_w} = \frac{\partial \alpha_0}{\partial \pi_T} + N^{-1} \frac{\partial \alpha_{\Delta w}}{\partial \pi_T} + O(N^{-2}) \quad \text{(see (3.4))},
\]

and using the definitions

\[
\begin{align*}
\hat{r}_{waa} & \equiv \frac{\partial \alpha_w}{\partial \pi_T} \Omega_T' \frac{\partial \alpha_w}{\partial \pi_T},
\hat{r}_{0aa} & \equiv \frac{\partial \alpha_0}{\partial \pi_T} \Omega_T' \frac{\partial \alpha_0}{\partial \pi_T},
\end{align*}
\]

(S.16)

and

\[
\hat{r}_{\Delta w0} = \frac{\partial \alpha_{\Delta w}}{\partial \pi_T} \Omega_T' \frac{\partial \alpha_0}{\partial \pi_T},
\]

we have

\[
\hat{r}_{waa}^{-1/2} = r_{waa}^{-1/2} - \frac{1}{2} r_{waa}^{-3/2} \frac{\partial \hat{r}_{wab}}{\partial \pi_T'} \left( p - \pi_T \right) + 3 \frac{3}{8} r_{waa}^{-5/2} \left( \frac{\partial \hat{r}_{wab}}{\partial \pi_T'} \right)^{<2>} \left( p - \pi_T \right)^{<2>}
\]

\[
- \frac{1}{4} r_{waa}^{-3/2} \frac{\partial^2 \hat{r}_{wab}}{(\partial \pi_T')}^{<2>} \left( p - \pi_T \right)^{<2>} + O_p(N^{-3/2}),
\]

(S.17)

where

\[
\begin{align*}
\hat{r}_{waa} & = r_{waa} + 2N^{-1} r_{\Delta w0} + O(N^{-2}),
\hat{r}_{waa}^{-1/2} & = r_{waa}^{-1/2} - N^{-1} r_{0aa}^{3/2} r_{\Delta w0} + O(N^{-2}),
\hat{r}_{waa}^{-j/2} & = r_{0aa}^{-j/2} + O(N^{-1}) \quad (j = 3, 5),
\end{align*}
\]

(S.18)

\[
\frac{\partial \hat{r}_{wab}}{\partial \pi_T} = \frac{\partial r_{0ab}}{\partial \pi_T} + O(N^{-1}), \quad \frac{\partial^2 \hat{r}_{wab}}{(\partial \pi_T)^{<2>}} = \frac{\partial^2 r_{0ab}}{(\partial \pi_T)^{<2>}} + O(N^{-1}).
\]

From the above results, (S.17) becomes
\[
= r_{0aa}^{-1/2} - N^{-1} r_{0aa}^{-3/2} r_{\Delta W_0} - \frac{N^{-1/2}}{2} r_{0aa}^{-3/2} \frac{\partial r_{0ab}}{\partial \pi_T} \mathbf{m}^{(1)}
\]
\[
+ N^{-1} \left\{ \frac{3}{8} r_{0aa}^{-5/2} \left( \frac{\partial r_{0aa}}{\partial \pi_T} \right)^{<2>} - \frac{1}{4} r_{0aa}^{-3/2} \frac{\partial^2 r_{0aa}}{\partial \pi_T^{<2>}} \right\} \mathbf{m}^{(1)<2>} + O_p(N^{-3/2}) \quad \text{(S.19)}
\]
\[
= r_{0aa}^{-1/2} + N^{-1} \eta_{R0} + \sum_{j=1}^{2} N^{-j/2} \kappa_R^{(j)} \mathbf{m}^{(j)} + O_p(N^{-3/2}),
\]
where \( \eta_{R0} = -r_{0aa}^{-3/2} r_{\Delta W_0}, \kappa_R^{(1)} = -\frac{1}{2} r_{0aa}^{-3/2} \frac{\partial r_{0aa}}{\partial \pi_T}, \)
and \( \kappa_R^{(2)} = \frac{3}{8} r_{0aa}^{-5/2} \left( \frac{\partial r_{0aa}}{\partial \pi_T} \right)^{<2>} - \frac{1}{4} r_{0aa}^{-3/2} \frac{\partial^2 r_{0aa}}{\partial \pi_T^{<2>}}. \)

In (S.19), using \( \frac{\partial \Omega_T}{\partial \pi_{Tj}} = \mathbf{E}_{jj} - \pi_T \mathbf{e}_j \mathbf{e}_j^\top \mathbf{e}_j \mathbf{e}_j^\top \) where \( \mathbf{e}_j \) is the vector whose \( j \)-th element is 1 with other ones being zero, we have
\[
\frac{\partial r_{0aa}}{\partial \pi_{Tj}} = 2 \frac{\partial^2 \alpha_0}{\partial \pi_T \partial \pi_T} \Omega_T \frac{\partial \alpha_0}{\partial \pi_T} + \left( \frac{\partial \alpha_0}{\partial \pi_T} \right)^2 - 2 \frac{\partial \alpha_0}{\partial \pi_T} \pi_T \frac{\partial \alpha_0}{\partial \pi_T}.
\]
\[
\frac{\partial^2 r_{0aa}}{\partial \pi_T \partial \pi_T} = 2 \frac{\partial^3 \alpha_0}{\partial \pi_T \partial \pi_T \partial \pi_T} \Omega_T \frac{\partial \alpha_0}{\partial \pi_T} + 2 \frac{\partial^2 \alpha_0}{\partial \pi_T \partial \pi_T} \Omega_T \frac{\partial^2 \alpha_0}{\partial \pi_T \partial \pi_T} \quad \text{(S.20)}
\]
\[
+ \sum_{(jk)}^2 2 \left( \frac{\partial^2 \alpha_0}{\partial \pi_T \partial \pi_T} \frac{\partial \alpha_0}{\partial \pi_T} - \frac{\partial^2 \alpha_0}{\partial \pi_T \partial \pi_T} \pi_T \frac{\partial \alpha_0}{\partial \pi_T} \right)
\]
\[
- 2 \frac{\partial \alpha_0}{\partial \pi_T} \frac{\partial \alpha_0}{\partial \pi_T} - 4 \frac{\partial^2 \alpha_0}{\partial \pi_T \partial \pi_T} \pi_T \frac{\partial \alpha_0}{\partial \pi_T} \quad (j, k = 1, \ldots, q).
\]
(e) \( \eta_{B-C V0} \)

The quantity \( \eta_{V0} \) is the generic expression for \( \eta_{i0} = (i_0^{(1)}, \eta_0) \), \( \eta_{H0} = (d_0^{(1B)}, \eta_0) \), \( \eta_{G0} = (g_0^{(1B)}, \eta_0) \) and

\[
\eta_{R0} = - r_{0aa}^{-3/2} r_{AW0}^{-1/2} \frac{\partial \alpha_{\Delta W}}{\partial \pi_T'}, \Omega_T \frac{\partial \alpha_0}{\partial \pi_T'} \].

The corresponding \( \eta_{B-C V0} \) is generically defined for \( \eta_{B-C i0} = (i_0^{(1)}, \eta_{B-C 0}) \), \( \eta_{B-C H0} = (d_0^{(1B)}, \eta_{B-C 0}) \), \( \eta_{B-C G0} = (g_0^{(1B)}, \eta_{B-C 0}) \) and \( \eta_{B-C R0} = r_{0aa}^{-3/2} \frac{\partial \beta_{ML1}}{\partial \pi_T'}, \Omega_T \frac{\partial \alpha_0}{\partial \pi_T'} \).

(f) A simplification of Theorem 6.

\( \kappa_2 (w_{B-P}^*) \) can be simplified as follows. Using (8.20), (8.19) becomes

\[
\frac{\partial \alpha_{B-P}}{\partial \pi_T'} = - \Lambda^{-1} \frac{\partial^2 I_{ML}}{\partial \alpha_0 \partial \pi_T'} - N^{-1} \frac{\partial \beta_{ML1}}{\partial \pi_T'} + O(N^{-2})
\]

\[
\equiv \frac{\partial \alpha_0}{\partial \pi_T'} + N^{-1} \frac{\partial \alpha_{AB-P}}{\partial \pi_T'} + O(N^{-2}),
\]

which gives

\[
\kappa_2 (w_{B-P}^*) = \beta_{ML2} + N^{-1} \left( \beta_{ML2} + 2 \frac{\partial \alpha_0}{\partial \pi_T'} \Omega_T \frac{\partial \alpha_{AB-P}}{\partial \pi_T'} \right) + O(N^{-2})\]

\[
\begin{align*}
\left( \beta_{B-P2} &= \beta_{ML2}, \right. \\
\beta_{B-P\Delta2} &= \beta_{B-CML\Delta2} = \beta_{ML\Delta2} - 2 \frac{\partial \alpha_0}{\partial \pi_T'} \Omega_T \frac{\partial \beta_{ML1}}{\partial \pi_T'} \right).
\end{align*}
\]

That is, we find that the asymptotic cumulants of \( \hat{\alpha}_{B-P} \) up to the fourth order and the higher-order asymptotic variance are identical to those of \( \hat{\alpha}_{B-CML} \) or the bias-corrected ML estimator, respectively.

Reference